Non-hyperbolicity of hyperkähler manifolds

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Plan of the talk

1. Introduce hyperkähler manifolds and their moduli. Define **the birational moduli space as a quotient of a Teichmüller space $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ by an arithmetic group Γ_I.**

2. Explore the non-Hausdorff properties of the birational moduli. Explain how the Moore's ergodic theorem is relevant.

3. Use the Brody's lemma to obtain non-hyperbolicity. Define twistor spaces and prove the Campana's theorem.

4. Prove non-hyperbolicity using Lagrangian fibrations.

Holomorphically symplectic manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A compact hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

Hilbert schemes

THEOREM: (a special case of Enriques-Kodaira classification) Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLES.

EXAMPLE: A Hilbert scheme of K3 is simple and hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known simple hyperkaehler manifolds are these 2 and two series:** Hilbert schemes of K3, and generalized Kummer.

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on M, and let $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

Remark: This terminology is **standard for curves.**

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta, \eta)^n$, for some primitive integer quadratic form q on $H^2(M, \mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \left(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb C} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb Z)$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb Z)}$ **is a finite index subgroup of $O(H^2(M,\mathbb Z),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb Z),q)$ **has finite kernel.**

The period map

Remark: For any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let P : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map P : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts) Two points $I, I' \in \operatorname{Teich}$ **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2.**

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient $\operatorname{Teich}_b/\Gamma$ is called **the birational moduli space** of M.

Monodromy group and the birational moduli space

THEOREM: Let (M,I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}\text{er}/\Gamma$, where $\mathbb{P}\text{er} = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$ and Γ is an arithmetic group in $O(H^2(M,\mathbb{R}),q)$**, called **the monodromy group**.

REMARK: Γ_I is a group generated by monodromy of the Gauss-Manin local system on $H^2(M)$.

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M,\mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

REMARK: Further on, **I shall freely identify $\mathbb{P}\text{er}$ and Teich_b.**

Ergodicity of the monodromy group action

The moduli space $\mathbb{P}\mathrm{er}/\Gamma_I$ **is extremely non-Hausdorff**.

THEOREM: (Calvin C. Moore, 1966) Let Γ be an arithmetic lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. Then the left action of Γ on G/H is **ergodic,** that is, **for all Γ-invariant measurable subsets $Z \subset G/H$, either Z has measure 0, or $G/H\backslash Z$ has measure 0.**

REMARK: This implies that **"almost all" Γ-orbits in G/H are dense**.

THEOREM: Let $\mathbb{P}\mathrm{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\mathrm{er}_e$ be a set of all points $L \subset \mathbb{P}\mathrm{er}$ such that the orbit $\Gamma \cdot L$ is dense. **Then $Z := \mathbb{P}\mathrm{er} \setminus \mathbb{P}\mathrm{er}_e$ has measure 0.**

Proof. Step 1: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ-action on G/H is ergodic,** by Moore's theorem.

Step 2: Ergodic orbits are dense, non-ergodic orbits have measure 0. ■

Ergodic complex structures

COROLLARY: For each hyperkähler manifold M **there exists a complex structure I such that any other complex structure I' in the same deformation class can be obtained as a limit of $\varphi_i I$,** where φ_i is a sequence of isotopies.

DEFINITION: We call a complex structure I **ergodic** if its orbit in $\mathbb{P}\mathrm{er}$ is dense.

PROBLEM: **Nobody has produced a concrete example of an ergodic complex structure** (so far).

Kobayashi hyperbolic manifolds

DEFINITION: **An entire curve** is a non-constant map $\mathbb{C} \longrightarrow M$.

DEFINITION: A compact complex manifold M is called **Kobayashi hyperbolic**, if there exist no entire curves $\mathbb{C} \longrightarrow M$.

THEOREM: **(Brody, 1975)**
Let I_i be a sequence of complex structures on M which are not hyperbolic, and I its limit. Then (M, I) is also not hyperbolic.

CONJECTURE: **All hyperkähler manifolds are non-hyperbolic.**

REMARK: **This conjecture would follow if we produce an ergodic complex structure which is non-hyperbolic.**

Twistor spaces and hyperkähler geometry

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata)

Rational curves on $\operatorname{Tw}(M)$**.**

DEFINITION: An ample rational curve on a complex manifold M is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$. It is called **a quasiline** if all $i_k = 1$.

CLAIM: Let M be a compact complex manifold containing a an ample rational line. **Then any N points $z_1, ..., z_N$ can be connected by an ample rational curve.**

CLAIM: Let M be a hyperkähler manifold, $\operatorname{Tw}(M) \stackrel{\sigma}{\longrightarrow} M$ its twistor space, $m \in M$ a point, and $S_m = \mathbb{C}P^1 \times \{m\}$ the corresponding rational curve in $\operatorname{Tw}(M)$. **Then S_m is a quasiline.**

Proof: Since the claim is essentially infinitesimal, it suffices to check it when M is flat. **Then** $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus 2p}) \cong \mathbb{C}P^{2p+1} \backslash \mathbb{C}P^{2p-1}$**, and S_m is a section of** $\mathcal{O}(1)^{\oplus 2p}$**.** ■

Entire curves in twistor fibers

THEOREM: (F. Campana)
Let M be a hyperkähler manifold, and $\mathrm{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ its twistor projection. **Then there exists an entire curve in some fiber of π.**

Proof: The space of rational curves in $\mathrm{Tw}(M)$ is not compact, because M is not Moishezon. Take a sequence $s_i : \mathbb{C}P^1 \longrightarrow \mathrm{Tw}(M)$ of rational curves which does not converge. Then $\lim_i |ds_i(I_i)| = \infty$ for some sequence $I_i \in \mathbb{C}P^1$. Take a subsequence for which I_i converges to some I. **Then $\pi^{-1}(I)$ contains an entire curve obtained as a limit of s_i, by Brody's theorem.** ■

COROLLARY: Let $N \subset \mathbb{P}\mathrm{er}$ be the set of all non-hyperbolic complex structures. **Then N contains a point on each rational curve $S \subset \mathbb{P}\mathrm{er}$ obtained from a hyperkähler structure.**

COROLLARY: N has Hausdorff codimension $\leqslant 2$.

REMARK: Such rational curves S correspond to 3-dimensional subspaces $W \subset H^2(M,\mathbb{R})$, with $\mathbb{P}\mathrm{er} = Gr_{+,+}(H^2(M,\mathbb{R}), S_W = Gr_{+,+}(W)$. **To prove non-hyperbolicity it would suffice to show that the set of ergodic points contains S_W for some $W \subset H^2(M,\mathbb{R})$.**

Divisors in the moduli space

Instead of taking a Γ-orbit of a point $s \in \mathbb{P}\mathrm{er}$, let's take an orbit of a subvariety.

DEFINITION: Given non-zero $\eta \in H^2(M, \mathbb{R})$, denote by $\mathbb{P}\mathrm{er}_\eta \subset \mathbb{P}\mathrm{er}$ the set of all $I \in \mathbb{P}\mathrm{er}$ such that $\eta \in H^{1,1}(M, I)$.

EXAMPLE: When $q(\eta,\eta) > 0$, and η is integer, η or $-\eta$ is ample when $\mathrm{Pic}(M, I) = \langle\eta\rangle$ (Huybrechts, Boucksom). The space $\mathbb{P}\mathrm{er}_\eta \subset \mathbb{P}\mathrm{er}$ is called **the polarized Teichmüller space** of M. It is a symmetric space. Its quotient $\mathbb{P}\mathrm{er}_\eta /\Gamma$ is quasiprojective, by Bailey-Borel's theorem, and Hausdorff.

THEOREM: (Anan′in-V.) For any integer η, **the quotient $\mathbb{P}\mathrm{er}_\eta /\Gamma$ is dense in the corresponding moduli space $\mathbb{P}\mathrm{er} /\Gamma$.**

For today's talk, we are interested in $q(\eta,\eta) = 0$.

Holomorphic Lagrangian fibrations

THEOREM: (Matsushita, 1997)
Let $\pi : M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold M to X, whith $0 < \dim X < \dim M$. **Then $\dim X = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian** (this means that the symplectic form vanishes on $\pi^{-1}(x)$).

DEFINITION: Such a map is called **holomorphic Lagrangian fibration**.

REMARK: The base of π is conjectured to be rational. Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

REMARK: The base of π has a natural flat connection on the smooth locus of π. The combinatorics of this connection can be used to determine the topology of M (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

REMARK: A manifold admitting a holomorphic Lagrangian fibration is non-hyperbolic, because it contains a torus.

The hyperkähler SYZ conjecture

CONJECTURE: (Tyurin, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon). Any hyperkähler manifold can be deformed to a manifold admitting a holomorphic Lagrangian fibration.

A trivial observation: Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration, and ω_X a Kähler class on X. **Then $\eta := \pi^* \omega_X$ is nef, and satisfies $q(\eta, \eta) = 0$.**

The hyperkähler SYZ conjecture: Let L be a nef line bundle on a hyperkähler manifold, with $q(L, L) = 0$. Then L is semiample. Here q is the Bogomolov-Beauville form.

THEOREM: (Kamenova-V.) Let $\eta \in H^2(M, \mathbb{Z})$ be a cohomology class satisfying $q(\eta, \eta) = 0$, and $I \in \mathbb{P}\text{er}_\eta$ a complex structure for which η is semiample. **Then η is semiample for a dense, open subset of $\mathbb{P}\text{er}_\eta$.**

Non-hyperbolicity of hyperkähler manifolds

COROLLARY: (Kamenova-V.) Let M be a hyperkähler manifold which has a deformation admitting a holomorphic Lagrangian fibration. **Then M is non-hyperbolic.**

Proof. Step 1: Let η be a nef class associated with a holomorphic Lagrangian fibration. Then η is semiample for a dense, open subset $\mathbb{P}\mathrm{er}_\eta^{sa} \subset \mathbb{P}\mathrm{er}_\eta$. **Since $\Gamma \cdot \mathbb{P}\mathrm{er}_\eta$ is dense in $\mathbb{P}\mathrm{er}$, $\Gamma \cdot \mathbb{P}\mathrm{er}_\eta^{sa}$ is also dense.**

Step 2: All points of $\Gamma \cdot \mathbb{P}\mathrm{er}_\eta^{sa}$ are non-hyperbolic, and the set $N \supset \Gamma \cdot \mathbb{P}\mathrm{er}_\eta^{sa}$ of non-hyperbolic points is closed in $\mathbb{P}\mathrm{er}$. Therefore, $N = \mathbb{P}\mathrm{er}$. ■

EXAMPLE: All known examples of hyperkähler manifolds (Hilbert schemes of K3, generalized Kummer, 2 of O'Grady's examples) have a deformation admitting a Lagrangian fibration. **Therefore, they are non-hyperbolic.**