Local structure of twistor spaces

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Plan

1. Hyperkähler manifolds. Twistor spaces.

2. Holography principle

3. Moishezon twistor spaces. Applications of the holography principle to the local structure of twistor spaces.

4. Proof of holography principle.

Twistor spaces (and hyperkähler geometry)

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

DEFINITION: **Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata)

Twistor spaces are non-Kähler

EXAMPLE: If $M = \mathbb{H}^n$, $\mathsf{Tw}(M) = \mathsf{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$ (total space of a vector bundle $(\mathcal{O}(1)^{\oplus n})$.

CLAIM: When M is compact, $\mathsf{Tw}(M)$ never admits a Kähler structure.

Proof: Let ω be the standard Hermitian form of $\mathsf{Tw}(M)$. Then $dd^c\omega$ is a positive (2,2)-form (a calculation due to Kaledin-V.) For any Kähler form ω_0, **this would imply**

$$\int_{\mathsf{Tw}(M)} d\left(\omega_0^{\dim_{\mathbb{C}} M-1} \wedge d^c\omega\right) = \int_{\mathsf{Tw}(M)} \omega_0^{\dim_{\mathbb{C}} M-1} \wedge dd^c\omega > 0,$$

which is impossible by Stokes' theorem. ■

Rational curves on $\operatorname{Tw}(M)$.

DEFINITION: **An ample rational curve** on a complex manifold M is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$. It is called **a quasiline** if all $i_k = 1$.

CLAIM: Let M be a compact complex manifold containing a an ample rational line. **Then any N points $z_1, ..., z_N$ can be connected by an ample rational curve.**

CLAIM: Let M be a hyperkähler manifold, $\operatorname{Tw}(M) \xrightarrow{\sigma} M$ its twistor space, $m \in M$ a point, and $S_m = \mathbb{C}P^1 \times \{m\}$ the corresponding rational curve in $\operatorname{Tw}(M)$. **Then S_m is a quasiline.**

Proof: Since the claim is essentially infinitesimal, it suffices to check it when M is flat. **Then $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus 2p}) \cong \mathbb{C}P^{2p+1} \backslash \mathbb{C}P^{2p-1}$, and S_m is a section of $\mathcal{O}(1)^{\oplus 2p}$.** ■

Holography principle

THEOREM: Let $S \subset M$ be a quasiline in a simply connected complex manifold, which is covered by deformations of S. Assume that M is equipped with a projection $\pi: M \longrightarrow S$ inducing identity on S. Consider a tubular neighbourhood $U \supset S$, and assume that $\pi: U \longrightarrow S$ has connected fibers. Then, **for any holomorphic line bundle L on M, the space $H^0(U, L)$ is independent from the choice of U and S in its deformation class.**

REMARK: In these assumptions, $H^0(U, L)$ **is always finite-dimensional** (the proof is further in these slides).

DEFINITION: Given a complex manifold Z, denote by $Mer(Z)$ the field of global meromorphic functions on Z, and let **the algebraic dimension** $k(Z)$ be the transcendence degree of $Mer(Z)$.

REMARK: It could be infinite!

COROLLARY: $k(\mathrm{Tw}(M)) = k(U)$**, for any tubular neighbourhood U of a quasiline,** satisfying the assumptions of the holography principle.

COROLLARY: Let M be a compact, simply connected hyperkähler manifold, and $U \subset M$ an open, connected subset. **Then $k(\mathrm{Tw}(U)) = 1$.**

Moishezon twistor spaces

DEFINITION: A compact complex variety Z is called **Moishezon** if the ring of meromorphic functions on Z has algebraic dimension $\dim Z$, or, equivalently, if Z is bimeromorphic to a projective manifold.

CLAIM: Let M be a simply connected hyperkaähler manifold, and $\operatorname{Tw}(M)$ its twistor space. **Then $k(\operatorname{Tw}(M)) \leqslant \dim_{\mathbb{C}} \operatorname{Tw}(M)$.**

DEFINITION: A twistor space for which it is equality is called **Moishezon**.

CLAIM: All Moishezon twistor spaces are bimeromorphic to open subsets of projective manifolds.

THEOREM: Let V be a quaterionic Hermitian vector space, and $G \subset \operatorname{Sp}(V)$ a compact Lie group acting on V by quaternionic isometries. Denote by M the hyperkähler reduction of V. **Then $\operatorname{Tw}(M)$ is Moishezon.**

Hamiltonians

Let's define the hyperkähler reduction.

We denote the Lie derivative along a vector field as $\operatorname{Lie}_x : \Lambda^i M \longrightarrow \Lambda^i M$, and contraction with a vector field by $i_x : \Lambda^i M \longrightarrow \Lambda^{i-1} M$.

Cartan's formula: $d \circ i_x + i_x \circ d = \operatorname{Lie}_x$.

REMARK: Let (M, ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms, and $\mathfrak{g}$ its Lie algebra. For any $g \in \mathfrak{g}$, denote by ρ_g the corresponding vector field. Then $\operatorname{Lie}_{\rho_g} \omega = 0$, giving $d(i_{\rho_g}(\omega)) = 0$. **We obtain that $i_{\rho_g}(\omega)$ is closed, for any $g \in \mathfrak{g}$.**

DEFINITION: A Hamiltonian of $g \in \mathfrak{g}$ is a function h on M such that $dh = i_{\rho_g}(\omega)$.

Moment maps

DEFINITION: (M, ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g \rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**.

DEFINITION: A G-invariant $c \in \mathfrak{g}^*$ is called **central**.

CLAIM: An equivariant moment map exists whenever $H^1(G, \mathfrak{g}^*) = 0$. In particular, when G is reductive and M is simply connected, an equivariant moment map exists.

Symplectic reduction and GIT

DEFINITION: (Weinstein-Marsden) (M,ω) be a symplectic manifold, G a compact Lie group acting on M by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of M, denoted by $M/\!\!/G$.

CLAIM: The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

THEOREM: Let (M,I,ω) be a Kähler manifold, $G_{\mathbb{C}}$ a complex reductive Lie group acting on M by holomorphic automorphisms, and G is compact form acting isometrically. **Then $M/\!\!/G$ is a Kähler orbifold.**

REMARK: In such a situation, $M/\!\!/G$ is called **the Kähler quotient**, or **GIT quotient**.

REMARK: The points of $M/\!\!/G$ are in bijective correspondence with the orbits of $G_{\mathbb{C}}$ which intersect $\mu^{-1}(c)$. Such orbits are called **(poly-)stable**, and the intersection of a $G_{\mathbb{C}}$-orbit with $\mu^{-1}(c)$ is a G-orbit.

Hyperkähler reduction

DEFINITION: Let G be a compact Lie group, ρ its action on a hyperkähler manifold M by hyperkähler isometries, and $\mathfrak{g}^*$ a dual space to its Lie algebra. **A hyperkähler moment map** is a G-equivariant smooth map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that $\langle \mu_i(v), g \rangle = \omega_i(v, d\rho(g))$, for every $v \in TM$, $g \in \mathfrak{g}$ and $i = 1, 2, 3$, where ω_i is one three Kähler forms associated with the hyperkähler structure.

DEFINITION: Let ξ_1, ξ_2, ξ_3 be three G-invariant vectors in $\mathfrak{g}^*$. The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of M.

THEOREM: (Hitchin, Karlhede, Lindström, Roček)
The quotient $M /\!\!/\!\!/ G$ is hyperkaehler.

Holomorphic moment map

Let $\Omega := \omega_J + \sqrt{-1}\omega_K$. This is a holomorphic symplectic (2,0)-form on (M, I).

The proof of HKLR theorem. Step 1: Let μ_J, μ_K be the moment map associated with ω_J, ω_K, and $\mu_{\mathbb{C}} := \mu_J + \sqrt{-1}\,\mu_K$. Then $\langle d\mu_{\mathbb{C}}, g\rangle = i_{\rho_g}(\Omega)$ Therefore, $d\mu_{\mathbb{C}} \in \Lambda^{1,0}(M, I) \otimes \mathfrak{g}^*$.

Step 2: This implies that the map $\mu_{\mathbb{C}}$ is holomorphic. It is called **the holomorphic moment map.**

Step 3: By definition, $M /\!\!/\!\!/ G = \mu_{\mathbb{C}}^{-1}(c) /\!\!/ G$, where $c \in \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$ is a central element. **This is a Kähler manifold,** because it is a Kähler quotient of a Kähler manifold.

Step 4: We obtain 3 complex structures I, J, K on the hyperkähler quotient $M /\!\!/\!\!/ G$. **They are compatible in the usual way** (an easy exercise). ■

Twistor spaces and hyperkähler reduction

THEOREM: Let V be a quaterionic Hermitian vector space, and $G \subset \operatorname{Sp}(V)$ a compact Lie group acting on V by quaternionic isometries. Denote by M the hyperkähler reduction of V. **Then** $\operatorname{Tw}(M)$ **is Moishezon.**

Proof: $\operatorname{Tw}(M)$ is obtained as the space of stable $G_{\mathbb{C}}$-orbits in $\mu_{\mathbb{C}}^{-1}(0) \subset \operatorname{Tw}(V)$. The space $\operatorname{Tw}(V) = \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$ is algebraic. Averaging over G, we obtain that the field $G_{\mathbb{C}}$-invariant rational functions on $\operatorname{Tw}(M)$ has dimension $\dim \operatorname{Tw}(M)$, and $\operatorname{Tw}(M)$ is Moishezon. ■

COROLLARY: Let U be an open subset of a compact, simply connected hyperkähler manifold, and U' an open subset of a hyperkähler manifold obtained as $V /\!/\!/ G$, where V is flat and G reductive. **Then U is not isomorphic to U' as hyperkähler manifold**.

Proof: $\operatorname{Tw}(U')$ has many meromorphic functions, and $\operatorname{Tw}(U)$ has very few. ■

$H^0(U, L)$ for a neighbourhood of an ample curve

PROPOSITION: Let $C \subset M$ be an ample rational curve, $U \supset C$ its tubular neighbourhood, and B a holomorphic bundle on U. **Then $H^0(U, B)$ is finite-dimensional.**

Proof. Step 1: Let $I_C \subset \mathcal{O}_U$ be the ideal sheaf of C, and $J_C^r(B) := B/I_C^{r+1}B$ the sheaf of r-jets of the sections of B. Since $I_C^r/I_C^{r+1} = \operatorname{Sym}^r(N^*C)$, the bundle $\operatorname{Sym}^r(N^*C)$ is a direct sum of $\mathcal{O}(k_i)$ with $k_i < -r$. **Therefore, $H^0(I_C^r B/I_C^{r+1}B) = 0$ for r sufficiently big.**

Step 2: $I_C^r B$ admits a filtration $I_C^r B \supset I_C^{r+1}B \supset I_C^{r+2}B \supset ...$ with associated graded sheaves $I_C^r B/I_C^{r+1}B$ having no sections. **Therefore, $H^0(I_C^r B) = 0$.**

Step 3: If $H^0(U, B)$ is infinite-dimensional, the map $H^0(U, B) \longrightarrow H^0(J_C^r(B))$ cannot be injective. Then, **for each r there exists a non-zero section with vanishing r-jet: $f_r \in H^0(U, I_C^{r+1}B)$.** This is impossible as shown in Step 2. ∎

Complex manifolds and quasilines

REMARK: Let $S \subset M$ be a quasiline. Then, for an appropriate tubular neighbourhood $U \subset M$ of S, **"for every two points $x, y \in U$ close to S and far from each other, there is a unique deformation of S containing X and Y."**

More precisely:

CLAIM: Let $S \subset M$ be a quasi-line. Then, for any sufficiently small tubular neighbourhood $U \subset M$ of S, there exists a smaller tubular neighbourhood $W \subset U$, satisfying the following condition. Let Δ_S be the image of the diagonal embedding $\Delta_S : S \longrightarrow W \times W$. Then there exists an open neighbourhood V of Δ_S, properly contained in $W \times W$, such that **for any pair $(x, y) \in W \times W \backslash V$, there exists a unique deformation $S' \subset U$ of S containing x and y.**

COROLLARY: For any quasiline in M, its deformation space is 2(dim $M-1$)-dimensional.

Proof of holography principle

THEOREM: (holography principle) Let $S \subset M$ be a quasiline in a simply connected complex manifold, which is covered by deformations of S. Assume that M is equipped with a projection $\pi : \ M \longrightarrow S$ inducing identity on S. Consider a tubular neighbourhood $U \supset S$, and assume that $\pi : \ U \longrightarrow S$ has connected fibers. Then, **for any holomorphic line bundle L on M, the space $H^0(U, L)$ is independent from the choice of U and S in its deformation class.**

A (slighly) weaker statement.

THEOREM 1: Let $S \subset M$ be a quasiline, and L a holomorphic bundle on M. Assume that M is equipped with a projection $\pi : \ M \longrightarrow S$ inducing identity on S. Consider a sufficiently small tubular neighbourhood $U \supset S$, and a smaller tubular neighbourhood $V \subset U$. **Then the restriction map $H^0(U, L) \longrightarrow H^0(V, L)$ is an isomorphism.**

We deduce the "Holography principle" from Theorem 1.

Deducing the holography principle from Theorem 1

Step 1: Choose a continuous, connected family S_b of quasilines parametrized by B such that $\bigcup_{b\in B} S_b = M$ Find a tubular neighbourhood U_b for each S_b in such a way that an intersection $U_b \cap U_{b'}$ for sufficiently close b, b' always contains S_b and $S_{b'}$. **By Theorem 1, $H^0(U_b \cap U_{b'}, L) = H^0(U_b, L) = H^0(U_{b'}, L)$.**

Step 2: Since B is connected, **all the spaces $H^0(U_b, L)$ are isomorphic,** and these isomorphisms are compatible with the restrictions to the intersections $U_b \cap U_{b'}$.

Step 3: Let now $f \in H^0(U_b, L)$, and let $\tilde{M}_f$ be the **domain of holomorphy** for f, that is, a maximal domain (non-ramified over M) such that f admits a holomorphic extension to $\tilde{M}_f$. Since $\cup U_b = M$, and f can be holomorphically extended to any U_b, the domain $\tilde{M}_f$ is a covering of M. **Now, "holography principle" follows, because M is simply connected.** ∎

Proof of Theorem 1

THEOREM 1: Let $S \subset M$ be a quasiline, and L a holomorphic bundle on M. Assume that M is equipped with a projection $\pi: M \longrightarrow S$ inducing identity on S. Consider a sufficiently small tubular neighbourhood $U \supset S$, and a smaller tubular neighbourhood $V \subset U$. **Then the restriction map $H^0(U, L) \longrightarrow H^0(V, L)$ is an isomorphism.**

Proof (for $L = \pi^* \mathcal{O}(i)$. Step 1: Fix a point $x_0 = \infty$ in $\mathbb{C}P^1$. A section of L is the same as meromorphic function having a pole of degree $\leqslant d$ at $\pi^{-1}(\infty)$.

Step 2: For each section $S_1 : \mathbb{C}P^1 \longrightarrow M$ of π, a degree d meromorphic function on S_1 is uniquely determined by its values at any $d+1$ points of S_1.

Step 3: Given a meromorphic function $f \in H^0(V, L)$ and a quasiline S_1 intersecting V in an open set, we can extend f to a meromorphic function $\tilde{f}$ on S_1 by computing its values at $d+1$ distinct points $z_1, ..., z_{d+1}$ of $S_1 \cap V$. Whenever S_1 is in V, this procedure gives $f\Big|_{S_1}$. **By analytic continuation, the values of $\tilde{f}(z)$ at any $z \in S_1$ are independent from the choice of z_i and S_1.**

Step 4: Therefore, **$\tilde{f}$ is a well-defined meromorphic function on the union V_1 of all quasilines intersecting V.** For U sufficiently small, V_1 contains U. Therefore, any $f \in H^0(V, L)$ can be extended to U. ■