Kähler threefolds without subvarieties

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(joint work with F. Campana and J.-P. Demailly)

Campana simple manifolds

REMARK: Recall that **an algebraic dimension** $a(M)$ of a compact complex manifold M is the transcendence degree of the field of global meromorphic functions on M. It is known to be bounded by $\dim M$, When $a(M) = \dim M$, M is called **Moishezon**. Moishezon manifolds are bimeromorphically equivalent to projective ones.

CLAIM: **Kähler Moishezon manifolds are projective** (Moishezon).

DEFINITION: **Campana simple manifold** is a manifold which is not a union of its proper complex subvarieties.

REMARK: For Campana simple manifolds, **algebraic dimension is 0.**

EXAMPLE: General deformation of a complex torus of dimension $\geqslant 2$ has no subvarieties.

EXAMPLE: **General deformation of a Hilbert scheme of a K3 has no subvarieties** (V., 1997).

EXAMPLE: **General deformation of a maximal holonomy hyperkähler manifold is Campana simple** (V., 1996).

Threefolds without subvarieties

CONJECTURE: (Campana)
Any Campana simple threefold is bimeromorphic to a torus. Any Campana simple manifold is bimeromorphic to a torus or a hyperkähler orbifold.

Theorem 1: (Campana-Demailly-V.)
Let M be a compact Kähler 3-dimensional manifold. Assume that M has no non-trivial closed complex subvarieties. **Then M is a complex torus.**

Calabi-Yau manifolds

DEFINITION: A Calabi-Yau manifold is a compact Kaehler manifold with topologically trivial canonical bundle.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

THEOREM: (Bogomolov's decomposition) Let M be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering $\tilde{M}$ of M which is a product of Kaehler manifolds of the following form:**

$$\tilde{M} = T \times M_1 \times ... \times M_i \times K_1 \times ... \times K_j,$$

with all M_i, K_i simply connected, T a torus, and $\mathcal{H}\mathrm{ol}(M_l) = Sp(n_l)$, $\mathcal{H}\mathrm{ol}(K_l) = SU(m_l)$

COROLLARY: Any Campana simple Calabi-Yau manifold **is either a torus or a simple hyperkähler manifold.**

REMARK: Theorem 1 follows immediately if we prove that the canonical bundle $\Lambda^{3,0}(M)$ is trivial, because there are no hyperkähler manifolds in dimension 3.

Subvarieties in a torus

THEOREM: Let Z be a subvariety of a torus T. Then **the canonical bundle of Z is base point free and globally generated** (that is, semiample). Moreover, **the corresponding map $\varphi: Z \longrightarrow \mathbb{P}H^0(Z, K_Z)^*$ is a non-trivial holomorphic map, unless Z is a subtorus of T.**

Proof. Step 1: The bundle $\Omega^d T$, $d = \dim Z$ is globally generated, hence restrictions of the sections of $\Omega^d T$ to Z don't have base points (common zeros), and produce a holomorphic map to $\mathbb{C}P^m$.

Step 2: The map φ is non-trivial unless the K_Z is a trivial bundle. In the latter case, the rank of restriction map $H^0(\Omega^d T) \longrightarrow H^0(Z, K_Z)$ is 1, that is, **all tangent spaces to Z are parallel in T,** and Z is a torus. ■

Albanese map for Campana simple manifolds

COROLLARY: Let M be a Campana simple manifold, such that $H^1(M) \neq 0$. **Then M is a torus.**

Proof: Consider the Albanese map $M \longrightarrow H^{1,0}(M)/H^1(M,\mathbb{Z})$. It is non-trivial; **since M is not covered by proper subvarieties, M is embedded to a torus T.** Then M is a subtorus of T, because the holomorphic map $\varphi: M \longrightarrow \mathbb{P}H^0(M, K_M)^*$ has to be trivial. ∎

REMARK: To prove that a Campana simple threefold M is a torus, **it would suffice ether to show that $K_M = \mathcal{O}_M$ or to show that $H^1(M,\mathbb{R}) \neq 0$.**

Brunella's theorem

DEFINITION: A line bundle is called **pseudoeffective** if it admits a singular metric with positive curvature current.

DEFINITION: A **1-dimensional holomorphic foliation** on a complex manifold is a coherent subsheaf $\mathcal{F} \subset TM$ of rank 1.

DEFINITION: A manifold M is called **uniruled** if it is a union of rational curves $C_z \subset M$.

DEFINITION: Let F be a coherent sheaf on a complex manifold, and F^{**} its double dual. Then F^{**} is called **reflexization** of F.

CLAIM: Reflexization of a rank one sheaf is always a line bundle.

THEOREM: (M. Brunella)
Let M be a compact Kähler manifold equipped with a 1-dimensional holomorphic foliation $\mathcal{F} \subset TM$, and $F := \mathcal{F}^{**}$ its reflexization. **Then either F^* is pseudoeffective, or M is uniruled.**

Brunella theorem for Campana simple threefolds

THEOREM: (Kodaira) Let M be a Kähler manifold, and $[\omega] \in H^2(M, \mathbb{R})$ its Kähler class. Suppose that ω is rational. **Then M is projective.**

COROLLARY: Let M be a non-projective compact Kähler manifold. **Then $H^{2,0}(M) \neq 0$,** in other words, M admits a holomorphic 2-form.

Proof: Assume that $H^{2,0}(M) = 0$. Then $H^{1,1}(M) = H^2(M)$. Since the Kähler cone is open in $H^{1,1}(M, \mathbb{R})$, it contains a rational point, hence M is projective. ■

THEOREM: Let M be a Campana simple 3-dimensional compact Kähler manifold. **Then its canonical bundle K_M is pseudoeffective.**

Proof: Consider a non-zero holomorphic 2-form $\Omega \in \Omega^2 M$, and let $\mathcal{F}$ be the kernel of Ω. Since the rank of Ω is 2, one has $\operatorname{rk} \mathcal{F} = 1$. **Brunella's theorem implies that F^* is pseudoeffective,** because otherwise M is uniruled. However, the normal bundle $TM/\mathcal{F}$ is equipped with a non-degenerate a 2-form Ω, hence its determinant vanishes. This gives $F^* = K_M$. ■

Big and nef currents

DEFINITION: Let M be a compact, n-dimensional complex manifold. A positive, closed (1,1)-current θ is called **nef** if it is a limit of positive, closed $(1,1)$-forms. If the cohomology class $[\theta]$ in addition satisfies $\int_M [\theta]^n > 0$, then θ is called **big**.

THEOREM: Let L be a holomorphic line bundle on a Kähler manifold with $c_1(L)$ represented by a big and nef class (such a bundle is called **big and nef**). **Then the function $P(N) := \dim H^0(L^N)$ grows as a polynomial of degree $\dim M$.**

Proof: This result follows from the Demailly's holomorphic Morse inequalities. ■

REMARK: Such a rate of growth guarantees that M is Moishezon. Then, **a Kähler manifold admitting a big and nef class is projective.**

COROLLARY: Therefore, **a Campana simple threefold has no big and nef bundles.**

Lelong numbers and Lelong sets

DEFINITION: Let T be a positive (p,p)-current on an n-dimensional manifold M, $z \in M$ a point, and d_z distance to z. Consider the current $dd^c \log d_z$. In a neighbourhood of z it is positive, and one can take the product $T \wedge (dd^c \log d_z)^{n-p}$. Its mass at z is called **the Lelong number of T at z**, denoted by $\nu_z(T)$. It measures how singular T is at Z.

DEFINITION: Let $c > 0$ be a number. **Lelong set** F_c of T is the set of z such that $\nu_z(T) \geqslant c$.

THEOREM: (Siu) **Lelong sets are complex analytic subvarieties of M.**

THEOREM: (Nadel) Let h be a singular metric on a line bundle L. Consider the sheaf L_{L^2} of L^2-integrable holomorphic sections of L. **Then L_{L^2} is a coherent subsheaf of L.**

DEFINITION: The **multiplier ideal** of h is an ideal I_h such that $I_h \otimes L = L_{L^2}$.

THEOREM: (Nadel) **Support of a multiplier ideal of a bundle L is a Lelong set of its curvature current.**

Demailly's regularization of currents

THEOREM: Any positive, closed (1,1)-current T on a Kähler manifold (M,ω) can be approximated by a sequence T_k of closed, real $(1,1)$-currents in the same cohomology class satisfying the following:

(A) $T_k + \delta_k\omega \geqslant 0$, where $\{\delta_k\} \longrightarrow 0$.

(B) T_k are smooth outside of a complex analytic subset $Z_k \subset M$, with $Z_1 \subset Z_2 \subset ...$

(C) Let T_0 be a smooth form cohomologous to T. Then $T_k = T_0 + dd^c\psi_k$, where ψ_k is a non-increasing sequence of almost plurisubharmonic functions converging to an almost plurisubharmonic ψ, which satisfies $dd^c\psi + T_0 = T$.

(D) Locally around Z_k, the functions ψ_k have logarithmic poles:

$$\psi_k = \lambda_k \log \sum |g_{k,l}|^2 + \tau_k,$$

where $g_{k,l}$ are holomorphic functions vanishing on Z_k,and τ_k is smooth.

(E) The Lelong numbers $\nu(T_k, x)$ of T_k are non-decreasing in k for any $x \in M$ and converge to $\nu(T, x)$.

REMARK: In layman's terms, **any positive, closed (1,1)-current can be approximated by a sequence of positive currents with logarithmic singularities** and their Lelong sets would also converge.

COROLLARY: In particular, **any current with empty Lelong sets is nef.**

Currents with isolated singularities

THEOREM: Let θ be a positive (1,1)-current on a complex n-manifold, with all Lelong sets of dimension 0. **Then θ is nef. If, in addition, its Lelong sets are non-empty, then θ is also big.**

Proof. Step 1: The nef condition follows directly from approximation. Indeed, θ is locally approximated by a sum of a smooth form θ_k and a dd^c-exact current $dd^c\psi_k$ with isolated logarithmic singularities which locally look like $dd^c \log|x_0 - x|$. Then $\theta_k + \max(\psi_k, -C_k)$ is non-singular and approximates θ, for an appropriate sequence $C_k \longrightarrow \infty$.

Step 2: The top product $(dd^c \log|x_0 - x|)^n$ is a positive volume form, giving $\int_M \theta^n > 0$. ■

Positive currents on 3-manifolds without subvarieties

COROLLARY: Let M be a Kähler manifold without non-trivial subvarieties. **Then any positive, closed (1,1)-current θ on M with integer cohomology class has empty Lelong sets.**

Proof: As shown above, θ is nef, has isolated Lelong sets, and is big unless they are empty. Then a holomorphic line bundle L with $c_1(L) = [\theta]$ is big and nef, making M projective. ■

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties. **Then the canonical bundle K_M is nef, not big, and admits a singular metric with all Lelong sets empty.**

Proof: By Brunella's theorem, K_M admits a singular metric with positive curvature current θ. Then θ is nef and has empty Lelong sets as shown above. ■

Demailly-Peternell-Schneider vanishing theorem

The following theorem was rediscovered several times during 1990-ies (Enoki, Takegoshi, Morougane). Its most general form (which we use) is due to Demailly, Peternell and Schneider.

THEOREM: Let (M, I, ω) be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, K its canonical bundle, and L a holomorphic line bundle on M equipped with a singular Hermitian metric h. Assume that the curvature Θ of L is a positive current on M, and denote by $\mathcal{I}(h)$ the corresponding multiplier ideal. Then **the wedge multiplication operator $\eta \longrightarrow \omega^i \wedge \eta$ induces a surjective map**

$$H^0(\Omega^{n-i}M \otimes L \otimes \mathcal{I}(h)) \xrightarrow{\omega^i \wedge \cdot} H^i(K \otimes L \otimes \mathcal{I}(h)).$$

Here ω is considered as an element in $H^1(\Omega^1 M)$, and multiplication by ω maps $H^k(\Omega^{n-l}M \otimes F)$ to $H^{k+1}(\Omega^{n-l+1}M \otimes F)$. ■

Demailly-Peternell-Schneider for threefolds without subvarieties

Demailly-Peternell-Schneider, applied to 3-manifolds without subvarieties, gives the following vanishing result.

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties, and K its canonical bundle. **Then the multiplication map**

$$H^0(\Omega^{n-i}M \otimes K^m) \xrightarrow{\omega^i \wedge \cdot} H^i(K^{m+1})$$

is surjective for any $m \geqslant 0$.

Proof: The bundle K^m is pseudoeffective, and all its Lelong numbers vanish. Then the multiplier ideals for K^m are trivial. ■

Bounds on Euler characteristic

CLAIM: Let M be a complex manifold without divisors, and B a holomorphic bundle on M. **Then** $\dim H^0(B) \leqslant \operatorname{rk} B$**.**

Proof: Otherwise there would be a non-constant linear relation between sections; being non-constant, it must vanish on a divisor. ■

PROPOSITION: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties, K its canonical bundle, and $A(m) := \chi(K^m)$ the holomorphic Euler characteristic of the pluricanonical bundle. **Then** $|A(m)| \leqslant 4$ **for any** $m \geqslant 1$**.**

Proof: From Demailly-Peternell-Schneider it follows that

$$\begin{aligned}-\dim H^0(\Omega^1 M \otimes K^{m-1}) - \dim H^0(\Omega^3 M \otimes K^{m-1}) \leqslant \\ \leqslant \chi(K^m) \leqslant \dim H^0(\Omega^0 M \otimes K^{m-1}) + \dim H^0(\Omega^2 M \otimes K^{m-1}).\end{aligned}$$

The corresponding bundles have $\operatorname{rk} \leqslant 4$. By the above statement, their spaces of sections are no more than 4-dimensional. ■

Riemann-Roch-Hirzebruch and its applications

THEOREM: (Riemann-Roch-Hirzebruch)
Let $\chi(M,E) := \sum_{i=0}^{\dim_{\mathbb{C}} M}(-1)^i \dim_{\mathbb{C}} H^i(M,E)$ be a holomorphic Euler characteristic of E. **Then for any complex threefold M, one has $\chi(M,K_M^d) = \frac{d+3d^3+2d^3}{12}c_1(K_M)^3 + \frac{(1-12d)}{24}c_1(M)c_2(M)$.** ■

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties, K_M its canonical bundle, and $A(m) := \chi(M,K_M^m)$. **Then $A(m) = const$.**

Proof: $A(m)$ is a polynomial by Riemann-Roch-Hirzebruch theorem, but it's bounded, as shown above, hence constant. ■

COROLLARY: Let M be a 3-dimensional compact Kähler manifold without non-trivial compact complex subvarieties. **Then $c_1(K_M)^3 = 0$, and $c_1(M)c_2(M) = 0$.**

Proof: $c_1(K_M)^3 = 0$, because K_M is nef and not big. Then Riemann-Roch-Hirzebruch formula gives $A(m) := \frac{(1-12\,m)}{24}c_1(M)c_2(M)$. Since $A(m)$ is bounded, this implies $c_1(M)c_2(M) = 0$, giving $A(m) = 0$. ■

Proof of Theorem 1

The main result (Theorem 1) follows if we prove that M is Calabi-Yau or $H^1(M, \mathbb{R}) \neq 0$.

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties. **Then K_M is trivial.**

Proof: From Riemann-Roch-Hirzebruch, we obtain $\chi(O) = 1 - h^1(O) + h^2(O) - h^0(K_M) = 0$. Since M admits a holomorphic 2-form, $h^2(O) \geqslant 1$, which gives $h^0(K_M) \geqslant 1$. **However, a section of K_M cannot vanish, because M has no divisors.** This implies that K_M is trivial. ■