Ergodic theory and symplectic invariants

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Geometric structures

DEFINITION: "Geometric structure" on a manifold is a collection of tensors satisfying a certain set of differential equations.

Let me give some examples.

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

DEFINITION: Symplectic form on a manifold is a non-degenerate differential 2-form ω satisfying $d\omega = 0$.

Teichmüller space of geometric structures

Let $\mathcal{C}$ be the set of all geometric structures of a given type, say, complex, or symplectic. We put topology of uniform convergence with all derivatives on $\mathcal{C}$. Let $\operatorname{Diff}_0(M)$ be the connected component of its diffeomorphism group $\operatorname{Diff}(M)$ (**the group of isotopies**).

DEFINITION: The quotient $\mathcal{C}/\operatorname{Diff}_0$ is called **Teichmüller space** of geometric strictures of this type.

DEFINITION: The group $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is called **the mapping class group** of M. It acts on Teich by homeomorphisms.

DEFINITION: The orbit space $\mathcal{C}/\operatorname{Diff} = \operatorname{Teich}/\Gamma$ is called **the moduli space** of geometric structure of this type.

Today I will describe Teich and Γ for some symplectic manifolds, and explain the **ergodicity of Γ-action.**

Teichmüller space for symplectic structures

DEFINITION: Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold M, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with C^∞-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.

DEFINITION: Consider the group of diffeomorphisms, denoted Diff or $\operatorname{Diff}(M)$ as a Frechet Lie group, and denote its connected component ("group of isotopies") by Diff_0. The quotient group $\Gamma := \operatorname{Diff}/\operatorname{Diff}_0$ is called **the mapping class group** of M.

DEFINITION: **Teichmüller space of symplectic structures on M** is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp}/\operatorname{Diff}_0$. The quotient $\operatorname{Teich}_s/\Gamma = \operatorname{Symp}/\operatorname{Diff}$, is called **the moduli space of symplectic structures.**

REMARK: In many cases Γ acts on Teich_s with dense orbits, hence **the moduli space is not always well defined.**

DEFINITION: Two symplectic structures are called **isotopic** if they lie in the same orbit of Diff_0, and **diffeomorphic** is they lie in the same orbit of Diff.

Moser's theorem

DEFINITION: Define **the period map** $\operatorname{Per} : \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ mapping a symplectic structure to its cohomology class.

THEOREM: (Moser, 1965)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per} : \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

Theorem 1: (Moser)
Let ω_t, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold S. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in t. **Then all ω_t are diffeomorphic.**

THEOREM: (Moser)
The **Teichmüler space** Teich_s **is a manifold** (possibly, non-Hausdorff), and the **period map** $\operatorname{Per} : \operatorname{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

Proof: The period map $P : \operatorname{Symp} \longrightarrow H^2(M, \mathbb{R})$ is a smooth submersion. Its fibers are submanifolds, hence locally path connected. By Theorem 1, orbits of Per locally coincide with the fibers of P. Therefore, Per is locally a diffeomorphism. ■

Non-Hausdorff points on symplectic Teichmüller space

Example of D. McDuff found in Salamon, Dietmar, *Uniqueness of symplectic structures*, Acta Math. Vietnam. 38 (2013), no. 1, 123-144.

Let $M = S^1 \times S^1 \times S^2 \times S^2$ with coordinates $\theta_1, \theta_2 \in S^1 \subset \mathbb{C}^*$ and $z_1, z_2 \in S^2$. Let $\varphi_{\theta,z}$ $\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rotation around the axis $z \in \mathbb{C}P^1$ by the angle θ. **Consider the diffeomorphism $\Psi: M \longrightarrow M$ mapping $(\theta_1, \theta_2, z_1, z_2)$ to $(\theta_1, \theta_2, z_1, \varphi_{\theta_1,z_1}(z_2))$.**

THEOREM: Let ω_λ be the product symplectic form on $M = T^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ obtained as a product of symplectic forms of volume 1, 1, λ on T^2, $\mathbb{C}P^1$, $\mathbb{C}P^1$. **The form $\Psi^*(\omega_1)$ is homologous, but not diffeomorphic to ω_1.** However, **the form $\Psi^*(\omega_\lambda)$ is diffeomorphic to ω_λ for any $\lambda \neq 1$.**

(D. McDuff, *Examples of symplectic structures*, Invent. Math. 89 (1987), 13-36.)

Ergodic group action

DEFINITION: Let (M, μ) be a space with finite measure, and G a group acting on M preserving μ. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Proof. Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

CLAIM: A group G acts on M ergodically **if and only if any L^2-integrable G-invariant function on M is constant almost everywhere.**

Mapping class group action on $\operatorname{Teich}_s(A)$

DEFINITION: Symplectic volume of a symplectic manifold (M,ω), $\dim_{\mathbb{R}} M = 2n$, is $\int_M \omega^n$. Fix a positive number A, and let $\operatorname{Teich}_s(A)$ be the Teichmüller space of symplectic forms with symplectic volume A.

REMARK: The mapping class group $\frac{\operatorname{Diff}}{\operatorname{Diff}_0}$ acts on $H^2(M)$ and on $\operatorname{Teich}_s(A)$ **Quite often, this group is arithmetic, and this action is ergodic.**

In this case, **all semicontinuous symplectic invariants, evaluated on dense orbits, depend only on the symplectic volume.**

Known cases: K3 surface, hyperkähler manifolds, tori $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$, $n > 1$.

Kähler manifolds

DEFINITION: A Riemannian metric g on a complex manifold (M, I) is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

REMARK: This is equivalent to $\nabla\omega = 0$, where ∇ is Levi-Civita connection.

Symplectic structures of Kähler type

DEFINITION: A symplectic structure is **of Kähler type** if it equal to a Kähler form of some Kähler structure.

REMARK: Not all symplectic structures are of Kähler type. However, **all known symplectic structures on a K3 and a torus are of Kähler type**; it is conjectured that all are. From now on, we shall tacitly assume that all symplectic structures we consider **are of Kähler type.**

THEOREM: (E. Amerik, V.) Let M be a K3 surface Then the period map $\operatorname{Per}: \operatorname{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is an open embedding on the set of all standard symplectic structures**, and **its image is the set of all cohomology classes** v **such that** $q(\omega,\omega) > 0$**,** where q is a quadratic form on cohomology defined below.

REMARK: A similar result **is proven for standard symplectic structures on a torus and on a hyperkähler manifold.**

Ergodic group action

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

DEFINITION: A lattice in a Lie group is a discrete subgroup $\Gamma \subset G$ such that G/Γ has finite volume with respect to Haar measure.

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact semisimple Lie subgroup. **Then the left action of Γ on G/H is ergodic.**

Ratner's theorem

EXAMPLE: By Borel and Harish-Chandra theorem, **an integer lattice in a simple Lie group has finite covolume.**

DEFINITION: Unipotent element in a Lie group $G \subset GL(V)$ is an exponent of a nilpotent element in its Lie algebra.

THEOREM: Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then **the closure of any Γ-orbit in G/H is an orbit of a Lie subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.**

EXAMPLE: Let V be a real vector space with integer lattice and a non-degenerate rational bilinear symmetric form of signature $(3, k)$, $k > 0$, $G := SO^+(V)$ a connected component of the isometry group, $H \subset G$ the stabiliser of a positive vector $v \in V$, $H \cong SO^+(2, k)$, and $\Gamma \subset G$ an integer lattice. Consider the quotient $\mathbb{P}\text{er} := G/H$. **Then the closure of $\Gamma \cdot J$ in G/H is an orbit of a closed Lie subgroup $S \subset G$ containing H.** Moreover, **S is the smallest rational subgroup with this property.**

REMARK: In this situation, **either v is proportional to a rational vector, or $S = G$.** Indeed, there are no intermediate subgroups $SO^+(2, k) \subsetneq S \subsetneq SO^+(3, k)$.

Ergodicity of mapping class group action

THEOREM: (V., 2009)
Let M be a maximal holonomy hyperkähler manifold. **Then the image of the mapping class group Γ in $O(H^2(M,\mathbb{Z}))$ has finite index.**

COROLLARY: Γ acts on $\operatorname{Teich}_s(A)$ with dense orbits.

Proof: Applying Moore's theorem to Γ inside $G = SO(H^2(M,\mathbb{R}),q)$ and H the stabilizer of $\omega \in H^2(M,\mathbb{R})$, we obtain that the action of Γ on $\operatorname{Teich}_s(A) \subset H^2(M,\mathbb{R})$ **is ergodic on forms with fixed volume,** hence has dense orbits.
■

THEOREM: Let M be a hyperkähler manifold, Γ its mapping class group, and Teich_s the Teichmüller space of symplectic structures of hyperkähler type. **Then the dense orbits correspond to irrational symplectic classes,** and rational symplectic classes have closed orbits.

Proof: Follows from Ratner's theorems on classification of ergodic measures.
■

COROLLARY: On a hyperkähler manifold or a compact torus of dimension $2i > 2$, **any semicontinuous invariant of symplectic structures is constant on irrational symplectic forms** of standard type and fixed volume.

Gromov Capacity

DEFINITION: Let M be a symplectic manifold. Define **Gromov capacity** $\mu(M)$ as the supremum of radii r, for all symplectic embeddings from a symplectic balls B_r to M.

DEFINITION: Define **symplectic volume** of a symplectic manifold (M, ω) as $\int_M \omega^{\frac{1}{2}\dim M}$.

REMARK: Gromov capacity is obviously bounded by the symplectic volumes: a manifold of Gromov capacity r has volume $\geq \operatorname{Vol}(B_r)$. However, **there are manifolds of infinite volume with finite Gromov capacity.**

THEOREM: (Gromov)
Consider **a symplectic cylinder** $C_r := \mathbb{R}^{2n-2} \times B_r$ with the product symplectic structure. Then the Gromov capacity of C_r is r.

REMARK: This result was used by Gromov to study symplectic packing in $\mathbb{C}P^2$. He found the packing constant for 2 equal balls in $\mathbb{C}P^2$.

Packing constants

DEFINITION: Let (K, ω_K) be a $2n$-dimensional symplectic manifold with finite volume, and (M, ω_M) a symplectic manifold. We assume that K admits a symplectic embedding to a bounded domain in $\mathbb{R}^{2n}$ with a flat symplectic structure. The corresponding **packing constant** is supremum of all ε such that $(K, \varepsilon\omega_K)$ admits a symplectic embedding to (M, ω_M). It is easy to see that **the packing constant is semicontinuous as a function of ω_M (Entov-V.)**

REMARK: Packing constant is a generalization of Gromov's symplectic capacity.

REMARK: Applying ergodicity to packing constants, we obtain that **these packing constant are universal,** that is, independent from the choice of an irrational symplectic structure as long as its volume stays constant. Indeed, the packing constants are semicontinuous as functions of ω, and any semicontinuous, MCG-invariant function is constant on dense orbits.

REMARK: Packing constants were computed explicitly when K is a union of symplectic balls, ellipsoids, and M is a torus or a hyperkähler manifold. In this situation, **the only obstruction to packing is the symplectic volume of M** (Entov-V.). For more exotic shapes, nothing is known, though everybody is sure that **for the hyperkähler manifolds and the tori the packing should be unobstructed, for any symplectic domain $K \subset \mathbb{R}^{2n}$.**

Maslov class of a Lagrangian submanifold

CLAIM: Let V be a symplectic vector space, and S the Grassmannian of Lagrangian subspaces in V. **Then $\pi_1(S) = \mathbb{Z}$.**

Proof: Introduce a Hermitian structure on $V = \mathbb{R}^{2n}$, identifying it with $\mathbb{C}^n$. Since $U(n)$ acts transitively on the Lagrangian Grassmanian, we have $S = \frac{U(n)}{SO(n)}$, and the exact sequence

$$0 \longrightarrow \pi_1(SO(n)) \longrightarrow \pi)_1(U(n)) \longrightarrow \pi_1(S) \longrightarrow \pi_1(SO(n)) = 0$$

immediately gives $\pi_1(S) = \mathbb{Z}$

DEFINITION: Let $L \subset M$ be a Lagrangian submanifold in a symplectic manifold. Its **Maslov class** is a homomorphism $\pi_1(M, L) \longrightarrow \mathbb{Z}$ which takes a 2-disk D with boundary in L, trivialized the principal $Sp(2n)$-bundle $TM|_D$, producing a map $\partial D \longrightarrow S$ and associates a map $\pi_1(\partial D) \longrightarrow \pi_1(S) = \mathbb{Z}$ to the corresponding path in the Lagrangian Grassmannian S. A **Maslov-zero Lagrangian submanifold** is a submanifold with vanishing Maslov class.

THEOREM: (Entov, V.) Let L be a Maslov-zero Lagrangian torus in a K3 surface or a symplectic torus M. **Then the fundamental class of L is non-zero and primitive** in $H^n(M, \mathbb{Z})$, where $2n = \dim_{\mathbb{R}} M$.

Maslov-zero Lagrangian tori in K3 and complex tori

THEOREM: (Entov, V.) Let L be a Maslov-zero Lagrangian torus in a K3 surface or a symplectic torus M. **Then the fundamental class $[L]$ of L is non-zero and primitive** in $H^n(M,\mathbb{Z})$, where $2n = \dim_{\mathbb{R}} M$.

Proof. Step 1: Fukaya has proved that a Maslov-zero Lagrangian torus L in T^{2n} is not displaceable, that is, its image by Hamiltonion isotopy will intersect L. When M is the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the "split symplectic form" $\sum dx_{2i-1} \wedge dx_{2i}$, Abouzaid and Smith showed that $[L]$ is primitive using Fukaya's theorem. Sheridan and Smith have shown that $[L]$ is primitive when M is a "mirror quartic", that is, a crepant resolution of a quotient of Fermat quartic in $\mathbb{C}P^3$ by $(\mathbb{Z}/4)^2$. They used homological Mirror Symmetry. We deduce our result from these two theorems, using the ergodic action.

Step 2: From Moser's theorem ("Theorem 1") it follows that a small deformation ω' of a symplectic structure ω can be trivialized in a neighbourhood of a Lagrangian submanifold $L \subset (M,\omega)$, as long as $\omega'|_L$ is cohomologous to zero. This implies that **a Lagrangian subvariety can be deformed together with a small deformation of a symplectic form.**

Maslov-zero Lagrangian tori in K3 and complex tori (2)

Step 3: Let Γ be the mapping class group, acting on the symplectic Teichmüller. Ratner's theorem implies that **a Γ-orbit of any irrational symplectic form is dense.** In particular, the set of split symplectic forms is dense in Teich_s on a torus. Given a Maslov-zero Lagrangian torus $L \subset (T^{2n}, \omega)$ with non-primitive fundamental class, we deform it to a symplectic torus with split symplectic form, obtaining a contradiction.

Step 4: When M is a K3 surface, we apply Sheridan-Smith theorem, and find a mirror quartic K3 with irrational symplectic form for which Sheridan-Smith theorem holds. The orbit of this symplectic structure in Teich_s is dense. **Then any Maslov-zero Lagrangian torus $L \subset (M, \omega)$ with non-primitive fundamental class deforms to a nearby fiber which is isomorphic to a mirror quartic K3,** bringing a contadiction. ■