Kuga-Satake map for arbitrary dimension

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Hodge structures

DEFINITION: Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb C} = V_{\mathbb R} \otimes_{\mathbb R} \mathbb C$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **integer Hodge structure** if one fixes an integer lattice $V_{\mathbb Q}$ or $V_{\mathbb Z}$ such that $V_{\mathbb R} = V_{\mathbb Q} \otimes_{\mathbb Q} \mathbb R$ or $V_{\mathbb R} = V_{\mathbb Z} \otimes_{\mathbb Z} \mathbb R$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of integer Hodge structures is a map which is $U(1)$-invariant and preserves the lattice.

DEFINITION: **Weak polarization** on a Hodge structrure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb Q}^* \otimes V_{\mathbb Q}^*$ (symmetric or antisymmetric depending on parity of w). It is called **polarization** if it is in addition satisfies $-(\sqrt{-1}\,)^{p-q} h(x, \overline{x}) > 0$ for each non-zero $x \in V^{p,q}$.

DEFINITION: **Period space** of (weakly polarized) Hodge structure with dimensions $\dim V^{p,q} = v_{pq}$ is the space of all decompositions $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ such that the above conditions are sattisfied.

REMARK: The period space, for odd weight, **is a complex manifold.** Indeed, the $V^{q,p}$ spaces for $q > p$ are determined by $V^{p,q}$ uniquely, hence the period space is an open subspace in the space of k-tuples of subspaces $V^{p,q} \subset V \otimes \mathbb C$, with $p < q$. For even weight, **the period space for (weakly) polarized Hodge structures is again a complex manifold**.

Hodge structures and homogeneous spaces

EXAMPLE: The **Hodge structure of weight 1** is a decomposition $V_{\mathbb C} = V^{1,0} \oplus V^{0,1}$, with $\overline{V^{1,0}} = V^{0,1}$. Clearly, the Hodge structures of weight 1 are in bijective correspondence with complex structures on V. Therefore, **the period space of Hodge structures of weight 1 is identified with $GL(2n,\mathbb R)/GL(n,\mathbb C)$.**

EXAMPLE: The **Hodge structure of K3 type** is a (weakly polarized) Hodge structure $V_{\mathbb C} = \bigoplus_{\substack{p+q=2\\ p,q\geqslant 0}} V^{p,q}$ of weight 2 with $\dim V^{2,0}=1$.

REMARK: The period space of weakly polarized Hodge structures of K3 type is identified with the quadric of lines $Q := \{l \in \mathbb P V_{\mathbb C} \ \ | \ \ h(l,l) = 0, h(l,\bar l) \neq 0\}$.

THEOREM: (Kuga-Satake)
Let Q be the space of weakly polarized Hodge structures of K3 type on (W,h). Then **there exists a vector space V equipped with $SO(W)$-action and an $SO(W)$-equivariant embedding from Q to the space of Hodge structures of weight 1 on V.**

Kuga-Satake embedding and Clifford modules

THEOREM: (Kuga-Satake) Let Q be the space of weakly polarized Hodge structures of K3 type on (W, h). Then **there exists a vector space V equipped with $SO(W)$-action and an $SO(W)$-equivariant embedding from Q to the space of Hodge structures of weight 1 on V.**

Proof. Step 1: For any Hodge structure of K3 type, the corresponding action of $\mathfrak{u}(1)$ is generated by a skew-symmetric matrix μ of rank 2, acting trivially on the orthogonal complement to a 2-dimensional plane $l = \langle \operatorname{Re}\Theta, \operatorname{im}\Theta\rangle$, where Θ is a generator of $V^{2,0}$, and acting as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on l.

Step 2: Let $\mathcal{C}\ell(W)$ be the Clifford algebra of W, and V a space with $\mathcal{C}\ell(W)$-action (such space is called **Clifford module**. Using the standard embedding $\mathfrak{so}(W) \subset \mathcal{C}\ell(W)$, we can consider μ as an element of $\mathcal{C}\ell(W)$. Then $\mu^2 = -1$ in the Clifford algebra, and this gives a complex structure on V. ∎

REMARK: Kuga and Satake were interested in **constructing an embedding of the symmetric spaces** associated with polarized Hodge structures of weight 1 and of K3 type.

Hyperkähler manifolds

DEFINITION: (E. Calabi, 1978)
Let (M, g) be a Riemannian manifold equipped with three complex structure operators $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\operatorname{Id}.$$

Suppose that I, J, K are Kähler. Then (M, I, J, K, g) is called **hyperkähler**.

REMARK: A hyperkähler manifold M is equipped with 3 symplectic forms ω_I, ω_J, ω_K. The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ **is a holomorphic symplectic 2-form on** (M, I). ■

THEOREM: (Calabi-Yau) Let M be a compact, holomorphically symplectic Kähler manifold. Then M **admits a hyperkähler metric,** which is uniquely determined by the cohomology class of its Kähler form ω_I.

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

Kuga-Satake construction in arbitrary dimension

REMARK: Let M be a hyperkahler manifold Kuga-Satake construction **gives an embedding from $H^2(M)$ to the second cohomology of a torus, compatible with the Hodge structure.** Indeed, W is embedded to $\Lambda^2(V)$, where V is a $\mathcal{C}l(W)$-module.

THEOREM: For any hyperkahler manifold M of complex dimension n, **there exists a torus T of dimension $n+l$ and an embedding of cohomology space $H^*(M) \mapsto H^{*+l}(T)$ which is compatible with the Hodge structures and the Poincare pairing.** Moreover, this embedding is compatible with an action of the Lie algebra generated by all Lefschetz $sl(2)$-triples on M.

REMARK: The corresponding map from the period space of M to the period space of T **coinsides with the Kuga-Satake map.**

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M,I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **of maximal holonomy**, or **simple**, or **IHS**, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several maximal holonomy hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be of maximal holonomy.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$,** for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n}\right)\left(\int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2 - 3, 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega \rangle$, where ω is a Kähler form.

Multi-dimensional BBF form

This is the original motivation for the present work.

1. BBF form is a weak polarization on $H^2(M)$ compatible with all complex structures.

2. It is not hard to produce a weak polarization on $H^k(M)$ compatible with all complex structures. However, there is no canonical choice.

DEFINITION: Let $a, b \in H^{2k}(M)$, M Kähler of complex dimension $2n$, and $q \in \operatorname{Sym}^2(H^2(M)) \subset H^4(M)$ be the element corresponding to the BBF form. Then **the multi-dimensional BBF form** is $a, b \longrightarrow \int_M a \wedge b \wedge q^{n-k}$.

CONJECTURE: It is non-degenerate.

Lie superalgebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i\in\mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\operatorname{End}(V^*) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}^i(V^*)$ is also graded, with $\operatorname{End}^i(V^*) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}(V^j, V^{i+j})$

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a, b\} = ab - (-1)^{\tilde{a}\tilde{b}} ba$.

DEFINITION: **A graded Lie algebra** (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \ \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a, b\} = -(-1)^{\tilde{a}\tilde{b}}\{b, a\}$ and satisfies **the super Jacobi identity** $\{c, \{a, b\}\} = \{\{c, a\}, b\} + (-1)^{\tilde{a}\tilde{c}}\{a, \{c, b\}\}$

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L(\alpha) := \omega \wedge \alpha$

2. $\Lambda(\alpha) := *L * \alpha$. It is easily seen that $\Lambda = L^*$.

3. The Weil operator $W\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension (5|4)**,** acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence **$\mathfrak{a}$ also acts on the cohomology of M.**

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$-action.

Supersymmetry in hyperkähler geometry

Let (M, I, J, K, g) be a hyperkaehler manifold, ω_I, ω_J, ω_K its Kaehler forms. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ, because it is Riemannian.

1. $L_I(\alpha) := \omega_I \wedge \alpha$

2. $\Lambda_I(\alpha) := *L_I * \alpha$. It is easily seen that $\Lambda_I = L_J^*$.

3. Three Weil operators $W_I\Big|_{\Lambda^{p,q}(M,I)} = \sqrt{-1}(p-q)$, $W_J\Big|_{\Lambda^{p,q}(M,J)} = \sqrt{-1}(p-q)$, $W_K\Big|_{\Lambda^{p,q}(M,K)} = \sqrt{-1}\,(p-q)$

THEOREM: These operators generate a Lie superalgebra $\mathfrak{a}$ of dimension $(11|8)$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in $\mathfrak{a}$, hence $\mathfrak{a}$ **also acts on the cohomology of M.**

REMARK: The Weil operators form the Lie algebra $\mathfrak{su}(2)$ of unitary quaternions. This means that **the quaternionic action belongs to $\mathfrak{a}$**. In particular, L_J, L_K, Λ_J and Λ_K.

REMARK: The twisted de Rham differentials d_I, d_J, d_K, associated to I, J, K also belong to $\mathfrak{a}$: $d_I = [W_I, d]$, $d_J = [W_J, d]$, $d_K = [W_K, d]$

$\mathfrak{so}(4,1)$-action and the Hodge decomposition

REMARK: 1. $[L_I, \Lambda_J] = W_K$, $[L_J, \Lambda_K] = W_I$, $[L_I, \Lambda_K] = -W_J$.

2. The even part of $\mathfrak{a}$ **is isomorphic to $\mathfrak{sp}(1,1,\mathbb{H}) \oplus \mathbb{R} \cdot \Delta$.**

3. The odd part $\langle d, d_I, d_J, d_K, d,^* d_I^*, d_J^*, d_K^* \rangle$ **generates the 9-dimensional odd Heisenberg algebra,** with the only non-trivial supercommutators being $\{d, d^*\} = \{d_I, d_I^*\} = \{d_J, d_J^*\} = \{d_K, d_K^*\} = \Delta$

4. The action of $\mathfrak{a}_{even}$ on $\mathfrak{a}_{odd}$ **is the fundamental representation of $\mathfrak{sp}(1,1,\mathbb{H})$ in $\mathbb{H}^2$,** with the quaternionic Hermitian metric on $\mathfrak{a}_{odd}$ provided by the anticommutator.

COROLLARY: The weight decomposition of the $\mathfrak{sp}(1,1,\mathbb{H}) = \mathfrak{so}(4,1)$-action on $H^*(M)$ **coincides with the Hodge decomposition.**

Lefschetz-Frobenius algebras

DEFINITION: A Frobenius algebra is a graded commutative algebra $A = \bigoplus_{i=0}^d A^i$ equipped with the Poincare-type non-degenerate product.

DEFINITION: A **Lefschetz triple** in a Frobenius algebra $A = \bigoplus_{i=0}^{2n} A^i$ is a triple of operators L_η, H, Λ_η where $\eta \in A^2$ is a fixed element, $L_\eta(x) := \eta \wedge x$, $H\Big|_{A^i} = i - n$ and Λ_η is an element such that L_η, H, Λ_η is an $\mathfrak{sl}(2)$-triple. A Frobenius algebra admitting a Lefschetz triple is called **a Lefschetz-Frobenius algebra** (Looijenga, Lunts).

REMARK: Such Λ_η **is uniquely determined by H and η** (this statement is sometimes called "Morozov's lemma", and sometimes included in the statement of Jacobson-Morozov theorem).

REMARK: Existence of Λ_η for given $\eta \in A^2$ is an open property in A^2, hence **a Lefschetz-Frobenius algebra admits many $\mathfrak{sl}(2)$-triples.**

Lia algebra $\mathfrak{g}$ generated by $\mathfrak{sl}(2)$-triples

THEOREM: Let M be a hyperkähler manifold of maximal holonomy, A^* its cohomology algebram and $\mathfrak{g} := \mathfrak{g}(A)$ the Lie algebra generated by all Lefschetz $\mathfrak{sl}(2)$-triples. **Then $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(b_2 - 2, 4)$.**

Sketch of the proof. Step 1: Consider the action of $\mathfrak{g}$ on the **Mukai extension** $\hat{H}^2(M) := \mathbb{R} \cdot x \oplus H^2(M) \oplus \mathbb{R} \cdot y$, where x has grading 0, y has grading 4, $H^2(M)$ has grading 2. We equip $\hat{H}^2(M)$ with **the Mukai form** which is equal to BBF on $H^2(M)$, preserves grading, and satisfies $q_M(x, y) = 1$ $x^2 = y^2 = 0$, $x, y \perp h^2(M)$ and $(x, y) = 1$. The action of $\mathfrak{g}$ on $\hat{H}^2(M)$ is determined by the following properties: **1. It is compatible with the grading. 2. For all $\alpha, \beta \in H^2(M)$, one has $L_\alpha x = \alpha$, $L_\alpha \beta = q(\alpha, \beta) y$, where q is the BBF form. 3. $\Lambda_\alpha y = \alpha$, $\Lambda_\alpha \beta = q(\alpha, \beta) x$.**

To see that this action is well-defined, we need to check that commutator relations hold. This follows from commutator relations in $\mathfrak{so}(1, 4)$ and Zariski density of pairs $\alpha, \beta \in \langle \omega_I, \omega_J, \omega_K \rangle$ in the set of all pairs $\alpha, \beta \in H^2(M)$.

Step 2: The map $\mathfrak{g} \longrightarrow \mathfrak{so}(\hat{H}^2(M))$ is surjective, which follows from the dimension argument (dimensions are computed using the local Torelli theorem). Injectivity of $\mathfrak{g} \longrightarrow \mathfrak{so}(\hat{H}^2(M))$ is clear, because $\mathfrak{so}(\hat{H}^2(M))$ is given by generators and relations which hold true in $\mathfrak{g}$. ■

Hodge structures and $\mathfrak{g}$-action

REMARK: The Lie algebra $\mathfrak{g} = \mathfrak{so}(b_1 - 2, 4)$ is equipped with a grading $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus g_2$, induced by the grading on the Mukai space: $\hat{H}^2(M) := H_0 \oplus H^2(M) \oplus H_4$, with H_0 and H_4 1-dimensional. Then $\mathfrak{g}_0 = \mathfrak{g}_0' \oplus H$, where $H = [L_\omega, \Lambda_\omega]$ is the operator inducing the grading and commuting with the rest of $\mathfrak{g}_0$, denoted by $\check{\mathfrak{g}}_0$.

REMARK: **The Lie algebra $\mathfrak{g}_0' := \mathfrak{so}(b_1 - 1, 3)$ is generated by the Weil maps W_I for all complex structures I of hyperkähler type obtained by deformations.** The corresponding Lie group G_0 acts as $\operatorname{Spin}(b_1 - 1, 3)$ in odd-dimensional cohomology and $SO(b_1 - 1, 3)$ on even-dimensional ones. It is generated by the complex structure action on $H^2(M)$ for all deformations of I.

COROLLARY: Let M be a hyperkähler manifold, and $H^*(M) \mapsto H^{*+l}(T)$ an embedding to the cohomology of a torus. Suppose that this embedding is compatible with an action of the Lie algebra generated by all Lefschetz $sl(2)$-triples on M. **Then it is compatible with the Hodge structures,** in the same sense as the usual Kuga-Satake map.

k-symplectic structures

DEFINITION: Let V be a $4n$-dimensional vector space, and $\Psi: W \longrightarrow \Lambda^2(V)$ a linear map. Assume that $\Psi(\omega)$ is a symplectic form for general $\omega \in W$, and has rank $\frac{1}{2}\dim W$ for ω in a non-degenerate quadric $Q \subset W$. Then Ψ is called **k-symplectic structure on V**, where $k = \dim W$.

EXAMPLE: For $k = 3$ this is **hypersymplectic structure**, which is the same as a triple of symplectic form $\omega_1, \omega_2, \omega_3$ such that $\omega_i \omega_j^{-1}$ act on V as the matrices algebra Mat(2). In a sense, hypersymplectic structure is "a complexification of a hyperkähler structure".

THEOREM: **(Soldatenkov-V.)** Let V be a k-symplectic space. **Then V is a Clifford module over a Clifford algebra $\mathcal{C}\ell(W)$.**

THEOREM: In this situation, **the algebra generated by $\mathfrak{sl}(2)$-triples associated with $\omega \in W$ acts on $\Lambda^*(V)$ as the algebra $\mathfrak{g} = \mathfrak{so}(\hat{W})$,** where $\hat{W} = W \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is Mukai extension of W.

Proof: We need to check that the $\mathfrak{so}(4,1)$-relations hold for any non-degenerate 3-dimensional subspace $W_3 \subset W$. However, this subspace is associated with a hypersymplectic structure, which is a complexification of a hyperkähler, and satisfies the same relations. ■

Proof of the main result

THEOREM: For any hyperkahler manifold M of complex dimension n, **there exists a torus T of dimension $n+k$ and an embedding of cohomology space $H^*(M) \mapsto H^{*+l}(T)$ which is compatible with the Hodge structures and the Poincare pairing.** Moreover, this embedding is compatible with an action of the Lie algebra generated by all Lefschetz $sl(2)$-triples on M.

Proof: Let $\mathfrak{g}$ be the Lie algebra generated by all $\mathfrak{sl}(2)$-triples, and $W := H^2(M)$. For any Clifford module V over $\mathcal{C}l(W)$, V admits a b_2-symplectic structure which gives $\mathfrak{g}$-action in $\Lambda^*(V)$. **If we manage to produce an embedding of $\mathfrak{g}$-modules $H^*(M) \hookrightarrow \Lambda^*(V)$, we are done.**

However, $\Lambda^*(V)$ is an exact representation of Spin$(\hat{W})$, hence its tensor powers contain any representation of Spin$(\hat{W})$. These tensor powers correspond to $\Lambda^*(V^n)$, which is also a Grassmann algebra for a Clifford module over $\mathcal{C}l(W)$.

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