

Kähler manifolds and Hodge theory

Misha Verbitsky

HSE, Nov. 10, 2011

Complex action on vector spaces

Let V be a vector space over \mathbb{R} , and $I : V \rightarrow V$ an automorphism which satisfies $I^2 = -\text{Id}_V$. **We extend the action of I on the tensor spaces $V \otimes V \otimes \dots \otimes V \otimes V^* \otimes V^* \otimes \dots \otimes V^*$ by multiplicativity:** $I(v_1 \otimes \dots \otimes w_1 \otimes \dots \otimes w_n) = I(v_1) \otimes \dots \otimes I(w_1) \otimes \dots \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues of I are $\pm\sqrt{-1}$.**
2. **V admits an I -invariant metric g .** Take any metric g_0 , and let $g := g_0 + I(g_0)$.
3. **I diagonalizable over \mathbb{C} .** Indeed, any orthogonal matrix is diagonalizable.
4. **All eigenvalues of I are equal to $\pm\sqrt{-1}$.** Indeed, $I^2 = -1$.
5. **There are as many $\sqrt{-1}$ -eigenvalues as there are $-\sqrt{-1}$ -eigenvalues.** Indeed, I is real.

The Hodge decomposition in linear algebra

DEFINITION: The Hodge decomposition $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$ -eigenspace of I , and $V^{0,1}$ a $-\sqrt{-1}$ -eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda_{\mathbb{R}}^n V \otimes_{\mathbb{R}} \mathbb{C}$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMARK: The operator I induces $U(1)$ -action on V by the formula $\rho(t)(v) = \cos t \cdot v + \sin t \cdot I(v)$. We extend this action on the tensor spaces by multiplicativity.

$U(1)$ -representations and the weight decomposition

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$** .

DEFINITION: A **weight decomposition** of a $U(1)$ -representation W is a decomposition $W = \bigoplus W^p$, where each $W^p = \bigoplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p .

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a **weight decomposition**, with $\Lambda^{p,q} V$ being a weight $p - q$ -component of $\Lambda^n V_{\mathbb{C}}$.

REMARK: $V^{p,p}$ is the space of $U(1)$ -invariant vectors in $\Lambda^{2p} V$.

Further on, TM is the tangent bundle on a manifold, and $\Lambda^i M$ the space of differential i -forms. It is a Grassman algebra on TM

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \rightarrow TM$ which satisfies $I^2 = -\text{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)

This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$ -linear map $N := \Lambda^2(T^{1,0}) \rightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I . **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Exercise: Prove that $\mathbb{C}P^n$ is a complex manifold, in the sense of the above definition.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g) .

REMARK: It is $U(1)$ -invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M , and ω **the Kähler form**.

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$ -invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$** . The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$ -invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$ -invariant 3-form on \mathbb{C}^n , but such a form must vanish, because $-\text{Id} \in U(n)$

REMARK: The same argument works for all symmetric spaces.

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler. Indeed, a restriction of a closed form is again closed.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g) .

REMARK: It is $U(1)$ -invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M , and ω **the Kähler form**.

Graded vector spaces and algebras

DEFINITION: A **graded vector space** is a space $V^* = \bigoplus_{i \in \mathbb{Z}} V^i$.

REMARK: If V^* is graded, the endomorphisms space $\text{End}(V^*) = \bigoplus_{i \in \mathbb{Z}} \text{End}^i(V^*)$ is also graded, with $\text{End}^i(V^*) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}(V^j, V^{i+j})$

DEFINITION: A **graded algebra** (or “graded associative algebra”) is an associative algebra $A^* = \bigoplus_{i \in \mathbb{Z}} A^i$, with the product compatible with the grading: $A^i \cdot A^j \subset A^{i+j}$.

REMARK: A bilinear map of graded spaces which satisfies $A^i \cdot A^j \subset A^{i+j}$ is called **graded**, or **compatible with grading**.

REMARK: The category of graded spaces can be defined as a **category of vector spaces with $U(1)$ -action**, with the weight decomposition providing the grading. Then **a graded algebra is an associative algebra in the category of spaces with $U(1)$ -action**.

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** \tilde{a} of an operator a is 0 if it is even, 1 if it is odd. We say that an operator is **pure** if it is even or odd.

Supercommutator

DEFINITION: A **supercommutator** of pure operators on a graded vector space is defined by a formula $\{a, b\} = ab - (-1)^{\tilde{a}\tilde{b}}ba$.

DEFINITION: A graded associative algebra is called **graded commutative** (or “supercommutative”) if its supercommutator vanishes.

EXAMPLE: The Grassmann algebra is supercommutative.

DEFINITION: A **graded Lie algebra** (Lie superalgebra) is a graded vector space \mathfrak{g}^* equipped with a bilinear graded map $\{\cdot, \cdot\} : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a, b\} = -(-1)^{\tilde{a}\tilde{b}}\{b, a\}$ and satisfies **the super Jacobi identity** $\{c, \{a, b\}\} = \{\{c, a\}, b\} + (-1)^{\tilde{a}\tilde{c}}\{a, \{c, b\}\}$

EXAMPLE: Consider the algebra $\text{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then $\text{End}(A^*), \{\cdot, \cdot\}$ is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d, d\} = 0$, and L an even or odd element. **Then $\{\{L, d\}, d\} = 0$.**

Proof: $0 = \{L, \{d, d\}\} = \{\{L, d\}, d\} + (-1)^{\tilde{L}}\{d, \{L, d\}\} = 2\{\{L, d\}, d\}$. ■

Hodge * operator

Let V be a vector space. **A metric g on V induces a natural metric on each of its tensor spaces:** $g(x_1 \otimes x_2 \otimes \dots \otimes x_k, x'_1 \otimes x'_2 \otimes \dots \otimes x'_k) = g(x_1, x'_1)g(x_2, x'_2)\dots g(x_k, x'_k)$.

This gives a natural positive definite scalar product on differential forms over a Riemannian manifold (M, g) : $g(\alpha, \beta) := \int_M g(\alpha, \beta) \text{Vol}_M$

Another non-degenerate form is provided by the **Poincare pairing**:
 $\alpha, \beta \longrightarrow \int_M \alpha \wedge \beta$.

DEFINITION: Let M be a Riemannian n -manifold. Define **the Hodge * operator** $*$: $\Lambda^k M \longrightarrow \Lambda^{n-k} M$ by the following relation: $g(\alpha, \beta) = \int_M \alpha \wedge * \beta$.

REMARK: The Hodge * operator always exists. It is defined explicitly in an orthonormal basis $\xi_1, \dots, \xi_n \in \Lambda^1 M$:

$$*(\xi_{i_1} \wedge \xi_{i_2} \wedge \dots \wedge \xi_{i_k}) = (-1)^s \xi_{j_1} \wedge \xi_{j_2} \wedge \dots \wedge \xi_{j_{n-k}},$$

where $\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_{n-k}}$ is a complementary set of vectors to $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_k}$, and s the signature of a permutation $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$.

REMARK: $*^2|_{\Lambda^k(M)} = (-1)^{k(n-k)} \text{Id}_{\Lambda^k(M)}$

Hodge theory

CLAIM: On a compact Riemannian n -manifold, one has $d^*|_{\Lambda^k M} = (-1)^{nk} *d*$, where d^* denotes **the adjoint operator**, which is defined by the equation $(d\alpha, \gamma) = (\alpha, d^*\gamma)$.

Proof: Since

$$0 = \int_M d(\alpha \wedge \beta) = \int_M d(\alpha) \wedge \beta + (-1)^{\tilde{\alpha}} \alpha \wedge d(\beta),$$

one has $(d\alpha, *\beta) = (-1)^{\tilde{\alpha}} (\alpha, *d\beta)$. Setting $\gamma := *\beta$, we obtain

$$(d\alpha, \gamma) = (-1)^{\tilde{\alpha}} (\alpha, *d(*)^{-1}\gamma) = (-1)^{\tilde{\alpha}} (-1)^{\tilde{\alpha}(\tilde{n}-\tilde{\alpha})} (\alpha, *d*\gamma) = (-1)^{\tilde{\alpha}\tilde{n}} (\alpha, *d*\gamma).$$

■

DEFINITION: The anticommutator $\Delta := \{d, d^*\} = dd^* + d^*d$ is called **the Laplacian** of M . It is self-adjoint and positive definite: $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$. Also, Δ commutes with d and d^* (Lemma 1).

THEOREM: (The main theorem of Hodge theory)

There is a basis in the Hilbert space $L^2(\Lambda^*(M))$ consisting of eigenvectors of Δ .

THEOREM: (“Elliptic regularity for Δ ”) Let $\alpha \in L^2(\Lambda^k(M))$ be an eigenvector of Δ . **Then α is a smooth k -form.**

De Rham cohomology

DEFINITION: The space $H^i(M) := \frac{\ker d|_{\Lambda^i M}}{d(\Lambda^{i-1} M)}$ is called **the de Rham cohomology of M** .

DEFINITION: A form α is called **harmonic** if $\Delta(\alpha) = 0$.

REMARK: Let α be a harmonic form. **Then** $(\Delta x, x) = (dx, dx) + (d^*x, d^*x)$, hence $\alpha \in \ker d \cap \ker d^*$

REMARK: The projection $\mathcal{H}^i(M) \longrightarrow H^i(M)$ from harmonic forms to cohomology is injective. Indeed, a form α lies in the kernel of such projection if $\alpha = d\beta$, but then $(\alpha, \alpha) = (\alpha, d\beta) = (d^*\alpha, \beta) = 0$.

THEOREM: **The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism** (see the next page).

REMARK: Poincare duality immediately follows from this theorem.

Hodge theory and the cohomology

THEOREM: The natural map $\mathcal{H}^i(M) \longrightarrow H^i(M)$ is an isomorphism.

Proof. Step 1: Since $d^2 = 0$ and $(d^*)^2 = 0$, one has $\{d, \Delta\} = 0$. This means that Δ commutes with the de Rham differential.

Step 2: Consider the eigenspace decomposition $\Lambda^*(M) \cong \bigoplus_{\alpha} \mathcal{H}_{\alpha}^*(M)$, where α runs through all eigenvalues of Δ , and $\mathcal{H}_{\alpha}^*(M)$ is the corresponding eigenspace. For each α , de Rham differential defines a complex

$$\mathcal{H}_{\alpha}^0(M) \xrightarrow{d} \mathcal{H}_{\alpha}^1(M) \xrightarrow{d} \mathcal{H}_{\alpha}^2(M) \xrightarrow{d} \dots$$

Step 3: On $\mathcal{H}_{\alpha}^*(M)$, one has $dd^* + d^*d = \alpha$. When $\alpha \neq 0$, and η closed, this implies $dd^*(\eta) + d^*d(\eta) = dd^*\eta = \alpha\eta$, hence $\eta = d\xi$, with $\xi := \alpha^{-1}d^*\eta$. This implies that the complexes $(\mathcal{H}_{\alpha}^*(M), d)$ don't contribute to cohomology.

Step 4: We have proven that

$$H^*(\Lambda^*M, d) = \bigoplus_{\alpha} H^*(\mathcal{H}_{\alpha}^*(M), d) = H^*(\mathcal{H}_0^*(M), d) = \mathcal{H}^*(M).$$

■

Supersymmetry in Kähler geometry

Let (M, I, g) be a Kähler manifold, ω its Kähler form. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ , because it is Riemannian.
1. $L(\alpha) := \omega \wedge \alpha$
2. $\Lambda(\alpha) := *L*\alpha$. It is easily seen that $\Lambda = L^*$.
3. The Weil operator $W|_{\Lambda^{p,q}(M)} = \sqrt{-1} (p - q)$

THEOREM: These operators generate a Lie superalgebra \mathfrak{a} of dimension $(5|4)$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in \mathfrak{a} , hence \mathfrak{a} also acts on the cohomology of M .

REMARK: This is a convenient way to summarize the Kähler relations and the Lefschetz' $\mathfrak{sl}(2)$ -action.

The coordinate operators

Let V be an even-dimensional real vector space equipped with a scalar product, and v_1, \dots, v_{2n} an orthonormal basis. Denote by $e_{v_i} : \Lambda^k V \rightarrow \Lambda^{k+1} V$ an operator of multiplication, $e_{v_i}(\eta) = e_i \wedge \eta$. Let $i_{v_i} : \Lambda^k V \rightarrow \Lambda^{k-1} V$ be an adjoint operator, $i_{v_i} = *e_{v_i}*$.

CLAIM: The operators e_{v_i} , i_{v_i} , Id are a basis of an **odd Heisenberg Lie superalgebra** \mathfrak{h} , with **the only non-trivial supercommutator given by the formula** $\{e_{v_i}, i_{v_j}\} = \delta_{i,j} \text{Id}$.

Now, consider the tensor $\omega = \sum_{i=1}^n v_{2i-1} \wedge v_{2i}$, and let $L(\alpha) = \omega \wedge \alpha$, and $\Lambda := L^*$ be the corresponding **Hodge operators**.

CLAIM: From the commutator relations in \mathfrak{h} , one obtains immediately that

$$H := [L, \Lambda] = \left[\sum e_{v_{2i-1}} e_{v_{2i}}, \sum i_{v_{2i-1}} i_{v_{2i}} \right] = \sum_{i=1}^{2n} e_{v_i} i_{v_i} - \sum_{i=1}^{2n} i_{v_i} e_{v_i},$$

is a scalar operator acting as $k - n$ on k -forms.

Integrability of the complex structure

CLAIM: (“Cartan’s formula”) The de Rham differential of can be expressed through the commutator of vector fields:

$$d\eta(X_1, \dots, X_{d+1}) = \sum (-1)^{i+1} D_{X_i}(\eta(X_1, \dots, \check{X}_i, \dots, X_{d+1})) \\ - \sum_{i < j} (-1)^{i+j+1} \eta([X_i, X_j], X_1, \dots, \check{X}_i, \dots, \check{X}_j, \dots, X_{d+1}).$$

For a 1-form η , this gives $d\eta(X_1, X_2) = D_{X_1}\eta(X_2) - D_{X_2}\eta(X_1) - \eta([X_1, X_2])$.

COROLLARY: Let (M, I) be an almost complex manifold. Then the following assertions are equivalent.

- (i) $d\eta \subset \Lambda^{0,2}(M) \oplus \Lambda^{1,1}(M)$ for any $\eta \in \Lambda^{0,1}(M)$.
- (ii) I is integrable.

REMARK: This is equivalent to $d|_{\Lambda^1 M}$ having only two Hodge components: $d = d^{1,0} + d^{0,1}$ (for a non-integrable complex structure, there are 4: $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$).

REMARK: Since $\Lambda^* M$ is multiplicatively generated by $\Lambda^1(M)$, the decomposition $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$ holds for any almost complex manifold.

Integrability and the Hodge decomposition

CLAIM: A manifold (M, I) is integrable if and only if $(d^{0,1})^2|_{C^\infty M} = 0$.

Proof. Step 1: The bundle $\Lambda^{1,0}(M)$ is generated over $C^\infty M$ by $d^{1,0}(C^\infty M)$. Indeed, it is n -dimensional, $n = \dim_{\mathbb{C}} M$ and to prove this one needs to find n functions f_1, \dots, f_n with $d^{1,0}f_i$ linearly independent at a point. This is done by taking $2n$ functions f_1, \dots, f_{2n} with df_i linearly independent, and finding an appropriate subset.

Step 2: Then, the integrability condition $d(\Lambda^{1,0}(M)) \subset \Lambda^{2,0}(M) \oplus \Lambda^{1,1}(M)$ is equivalent to $dd^{1,0}(C^\infty M) \subset \Lambda^{2,0}(M) \oplus \Lambda^{1,1}(M) \Leftrightarrow d^{-1,2}(d^{1,0}(C^\infty M)) = 0$.

Step 3: The $(0,2)$ component of $d^2 = 0$ gives $\{d^{-1,2}, d^{1,0}\} = \{d^{0,1}, d^{0,1}\} = 2(d^{0,1})^2 = 0$. From Step 2, we obtain that $(d^{0,1})^2|_{C^\infty M} = 0$ is equivalent to integrability. ■

REMARK: The above claim provides an equivalence $d^{2,-1} = 0 \Leftrightarrow \{d^{-1,2}, d^{1,0}\} = 0 \Leftrightarrow (d^{0,1})^2 = 0$.

The twisted differential d^c

DEFINITION: The **twisted differential** is defined as $d^c := IdI^{-1}$.

CLAIM: Let (M, I) be a complex manifold. **Then** $\partial := \frac{d + \sqrt{-1}d^c}{2}$, $\bar{\partial} := \frac{d - \sqrt{-1}d^c}{2}$ **are the Hodge components of d** , $\partial = d^{1,0}$, $\bar{\partial} = d^{0,1}$.

Proof: Let V be a space generated by d, IdI . The natural action of $U(1)$ generated by $e^{\mathcal{W}}$ preserves V . **Since d has only two Hodge components. $U(1)$ acts with weights $\sqrt{-1}$ and $-\sqrt{-1}$** , and its Hodge components are expressed as above. ■

CLAIM: On a complex manifold, one has $d^c = [\mathcal{W}, d]$.

Proof: Clearly, $[\mathcal{W}, d^{1,0}] = \sqrt{-1}d^{1,0}$ and $[\mathcal{W}, d^{0,1}] = -\sqrt{-1}d^{0,1}$. Adding these equations, obtain $d^c = [\mathcal{W}, d]$.

COROLLARY: $\{d, d^c\} = \{d, \{d, \mathcal{W}\}\} = 0$ (Lemma 1).

De Rham differential on Kaehler manifolds

THEOREM: The following statements are equivalent.

1. I is integrable.
2. $\partial^2 = 0$.
3. $\bar{\partial}^2 = 0$.
4. $dd^c = -d^c d$
5. $dd^c = 2\sqrt{-1} \partial\bar{\partial}$.

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

THEOREM: Let M be a Kaehler manifold. One has the following identities (“Kähler identities”).

$$[\Lambda, \partial] = \sqrt{-1} \bar{\partial}^*, \quad [L, \bar{\partial}] = -\sqrt{-1} \partial^*, \quad [\Lambda, \bar{\partial}^*] = -\sqrt{-1} \partial, \quad [L, \partial^*] = \sqrt{-1} \bar{\partial}.$$

Equivalently,

$$[\Lambda, d] = (d^c)^*, \quad [L, d^*] = -d^c, \quad [\Lambda, d^c] = -d^*, \quad [L, (d^c)^*] = d.$$

Laplacians and supercommutators

THEOREM: Let

$$\Delta_d := \{d, d^*\}, \quad \Delta_{d^c} := \{d^c, d^{c*}\}, \quad \Delta_\partial := \{\partial, \partial^*\}, \quad \Delta_{\bar{\partial}} := \{\bar{\partial}, \bar{\partial}^*\}.$$

Then $\Delta_d = \Delta_{d^c} = 2\Delta_\partial = 2\Delta_{\bar{\partial}}$. In particular, Δ_d **preserves the Hodge decomposition.**

Proof: By Kodaira relations, $\{d, d^c\} = 0$. Graded Jacobi identity gives

$$\{d, d^*\} = -\{d, \{\Lambda, d^c\}\} = \{\{\Lambda, d\}, d^c\} = \{d^c, d^{c*}\}.$$

Same calculation with $\partial, \bar{\partial}$ gives $\Delta_\partial = \Delta_{\bar{\partial}}$. Also, $\{\partial, \bar{\partial}^*\} = \sqrt{-1} \{\partial, \{\Lambda, \partial\}\} = 0$, (Lemma 1), and the same argument implies that **all anticommutators $\partial, \bar{\partial}^*$, etc. all vanish except $\{\partial, \partial^*\}$ and $\{\bar{\partial}, \bar{\partial}^*\}$.** This gives $\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}}$.

■

DEFINITION: The operator $\Delta := \Delta_d$ is called **the Laplacian**.

REMARK: We have proved that **operators $L, \Lambda, d, \mathcal{W}$ generate a Lie superalgebra of dimension $(5|4)$ (5 even, 4 odd), with a 1-dimensional center $\mathbb{R}\Delta$.**

The Lefschetz $\mathfrak{sl}(2)$ -action

COROLLARY: The operators L, Λ, H form a basis of a Lie algebra isomorphic to $\mathfrak{sl}(2)$, with relations

$$[L, \Lambda] = H, \quad [H, L] = 2L, \quad [H, \Lambda] = -2\Lambda.$$

DEFINITION: L, Λ, H is called **the Lefschetz $\mathfrak{sl}(2)$ -triple**.

REMARK: Finite-dimensional representations of $\mathfrak{sl}(2)$ are semisimple.

REMARK: A simple finite-dimensional representation V of $\mathfrak{sl}(2)$ is generated by $v \in V$ which satisfies $\Lambda(v) = 0$, $H(v) = pv$ (“**lowest weight vector**”), where $p \in \mathbb{Z}^{\geq 0}$. Then $v, L(v), L^2(v), \dots, L^p(v)$ form a basis of $V_p := V$. **This representation is determined uniquely by p .**

REMARK: In this basis, **H acts diagonally:** $H(L^i(v)) = (2i - p)L^i(v)$.

REMARK: One has $V_p = \text{Sym}^p V_1$, where V_1 is a 2-dimensional tautological representation. It is called **a weight p representation of $\mathfrak{sl}(2)$** .

COROLLARY: For a finite-dimensional representation V of $\mathfrak{sl}(2)$, denote by $V^{(i)}$ the eigenspaces of H , with $H|_{V^{(i)}} = i$. **Then L^i induces an isomorphism**

$$V^{(-i)} \xrightarrow{L^i} V^{(i)} \text{ for any } i > 0.$$

Lefschetz action on cohomology.

From the supersymmetry theorem, the following result follows.

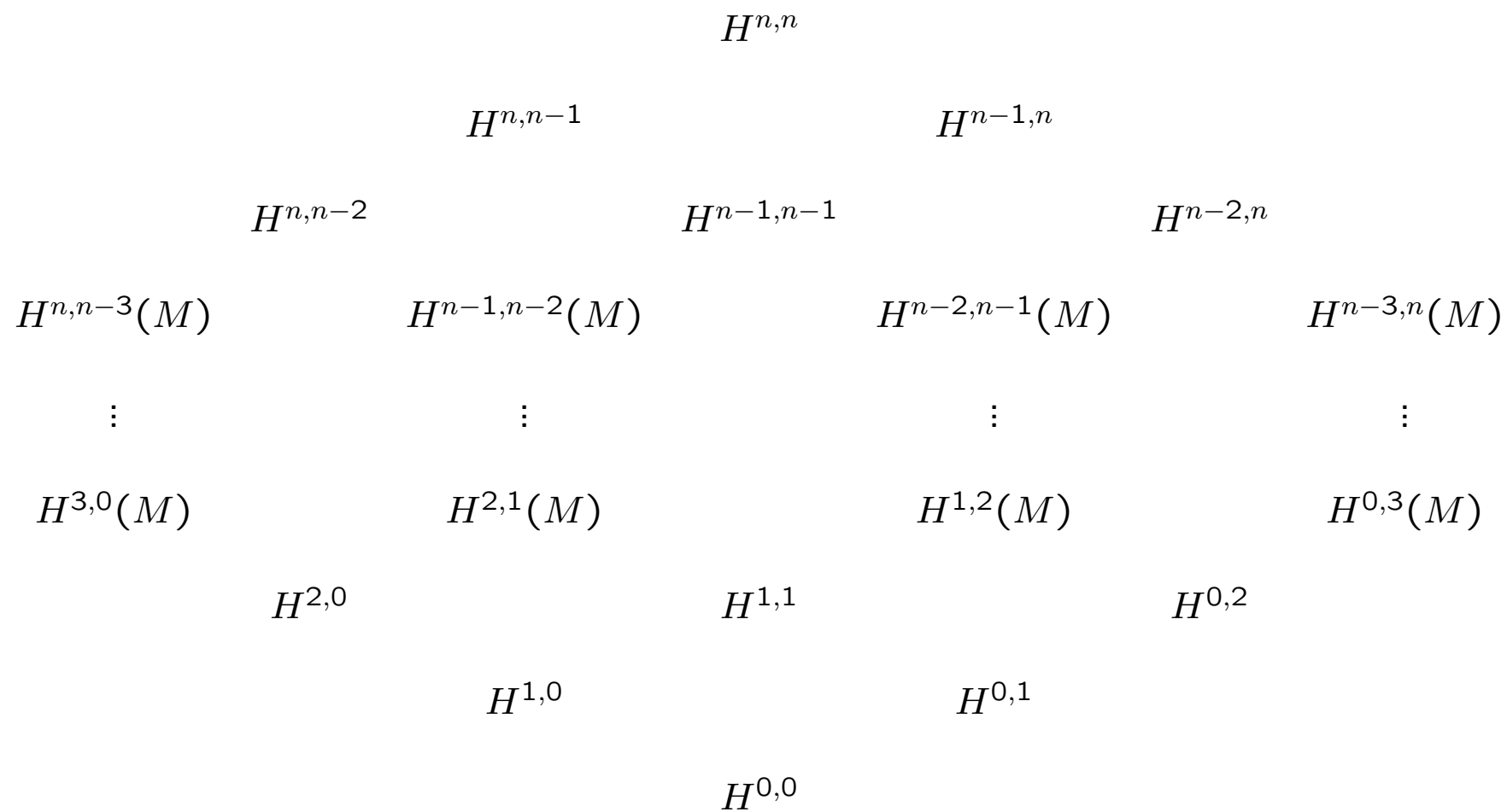
COROLLARY: The $\mathfrak{sl}(2)$ -action $\langle L, \Lambda, H \rangle$ and the action of Weil operator commute with Laplacian, hence **preserve the harmonic forms on a Kähler manifold.**

COROLLARY: Any cohomology class can be represented as a sum of closed (p, q) -forms, giving a decomposition $H^i(M) = \bigoplus_{p+q=i} H^{p,q}(M)$, with $\overline{H^{p,q}(M)} = H^{q,p}(M)$.

COROLLARY: odd cohomology of a compact Kähler manifold are even-dimensional.

COROLLARY: Let M be a compact, Kähler manifold of complex dimension n , and $i + p + q = n$. Then L^i defines **the Lefschetz isomorphism** $H^{p,q} \xrightarrow{L^i} H^{p+2i, q+2i}(M)$

The Hodge diamond:



Hyperkähler manifolds

DEFINITION: (E. Calabi, 1978)

Let (M, g) be a Riemannian manifold equipped with three complex structure operators $I, J, K : TM \rightarrow TM$, satisfying the quaternionic relation

$$I^2 = J^2 = K^2 = IJK = -\text{Id}.$$

Suppose that I, J, K are Kähler. Then (M, I, J, K, g) is called **hyperkähler**.

REMARK: A hyperkähler manifold M is equipped with 3 symplectic forms $\omega_I, \omega_J, \omega_K$. The form $\Omega := \omega_J + \sqrt{-1}\omega_K$ **is a holomorphic symplectic 2-form on (M, I)** . ■

THEOREM: (Calabi-Yau) Let M be a compact, holomorphically symplectic Kähler manifold. Then M **admits a hyperkähler metric**, which is uniquely determined by the cohomology class of its Kähler form ω_I .

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

Supersymmetry in hyperkähler geometry

Let (M, I, J, K, g) be a hyperkaehler manifold, $\omega_I, \omega_J, \omega_K$ its Kaehler forms. **On $\Lambda^*(M)$, the following operators are defined.**

0. d, d^*, Δ , because it is Riemannian.
1. $L_I(\alpha) := \omega_I \wedge \alpha$
2. $\Lambda_I(\alpha) := *L_I * \alpha$. It is easily seen that $\Lambda_I = L_J^*$.
3. Three Weil operators $W_I|_{\Lambda^{p,q}(M,I)} = \sqrt{-1}(p-q)$, $W_J|_{\Lambda^{p,q}(M,J)} = \sqrt{-1}(p-q)$, $W_K|_{\Lambda^{p,q}(M,K)} = \sqrt{-1}(p-q)$

THEOREM: These operators generate a Lie superalgebra \mathfrak{a} of dimension $(11|8)$, acting on $\Lambda^*(M)$. Moreover, the Laplacian Δ is central in \mathfrak{a} , hence \mathfrak{a} also acts on the cohomology of M .

REMARK: The Weil operators form the Lie algebra $\mathfrak{su}(2)$ of unitary quaternions. This means that **the quaternionic action belongs to \mathfrak{a}** . In particular, L_J, L_K, Λ_J and Λ_K .

REMARK: The twisted de Rham differentials d_I, d_J, d_K , associated to I, J, K also belong to \mathfrak{a} : $d_I = [W_I, d]$, $d_J = [W_J, d]$, $d_K = [W_K, d]$

Supersymmetry and the Hodge decomposition

REMARK: 1. $[L_I, \Lambda_J] = W_K$, $[L_J, \Lambda_K] = W_I$, $[L_I, \Lambda_K] = -W_J$.

2. The even part of \mathfrak{a} **is isomorphic to** $\mathfrak{sp}(1, 1, \mathbb{H}) \oplus \mathbb{R} \cdot \Delta$.

3. The odd part $\langle d, d_I, d_J, d_K, d, {}^* d_I^*, d_J^*, d_K^* \rangle$ **generates the 9-dimensional odd Heisenberg algebra**, with the only non-trivial supercommutators being $\{d, d^*\} = \{d_I, d_I^*\} = \{d_J, d_J^*\} = \{d_K, d_K^*\} = \Delta$

4. The action of $\mathfrak{a}_{\text{even}}$ on $\mathfrak{a}_{\text{odd}}$ **is the fundamental representation of** $\mathfrak{sp}(1, 1, \mathbb{H})$ **in** \mathbb{H}^2 , with the quaternionic Hermitian metric on $\mathfrak{a}_{\text{odd}}$ provided by the anticommutator.

REMARK: The weight decomposition of the $\mathfrak{sp}(1, 1, \mathbb{H}) = \mathfrak{so}(1, 4)$ -action on $H^*(M)$ **coincides with the Hodge decomposition.**