Exotic hypercomplex structures on complex tori

Misha Verbitsky

Estruturas geométricas em variedades,

April 3, 2025, IMPA

a joint work in progress with Alberto Pipitone.

HYPERCOMPLEX MANIFOLDS

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

THEOREM: (M. Obata, 1952)
Let (M, I, J, K) be a hypercomplex manifold. **Then M admits a unique torsion-free affine connection preserving I, J, K.**

REMARK: Converse is also true. Suppose that I, J, K are operators defining quaternionic structure on TM, and ∇ a torsion-free, affine connection preserving I, J, K. **Then I, J, K are integrable almost complex structures, and (M, I, J, K) is hypercomplex.**

Holonomy of Obata connection lies in $GL(n, \mathbb{H})$. Conversely, **a manifold equipped with an affine, torsion-free connection with holonomy in $GL(n, \mathbb{H})$ is hypercomplex**.

This can be used as a definition of a hypercomplex structure: **a hypercomplex manifold** (M, ∇, I, J, K) is a manifold equipped with a torsion-free connection such that its holonomy preserves a quaternionic structure on a tangent bundle.

Exotic hypercomplex structures on hyperkähler manifolds

DEFINITION: A hypercomplex manifold (M, ∇, I, J, K) is called **hyperkähler** if the holonomy $\mathcal{H}ol(\nabla)$ of ∇ is compact. In this case, $\mathcal{H}ol(\nabla)$ preserves a quaternionic invariant Riemannian metric g. Such metric is called **hyperkähler**. A **hyperkähler structure** is (M, ∇, I, J, K, g); in this situation, **∇ is the Levi-Civita connection**.

THEOREM: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that (M, I) admits a Kähler structure **Then (M, I) admits a hyperkähler structure (I, J', K').**

DEFINITION: Let (M, I, J, K) be a compact hypercomplex manifold. Assume that (M, I) admits a Kähler structure. The hypercomplex structure (I, J, K) is called **exotic** if it is not compatible with a hyperkähler metric, that is, if the holonomy of its Obata connection is non-compact.

Exotic hypercomplex structures on K3

THEOREM: Exotic hypercomplex structures on K3 do not exist.

Proof. **Step 1:** Let (M, I, J, K) be a hypercomplex structure on a K3, and Θ the curvature of Obata connection on its canonical bundle $K_{M,I} = K_{M,J} = K_{M,K}$. Since Θ is of type (1,1) for I, J, K, it is $SU(2)$-invariant with respect to the $SU(2)$-action on $\Lambda^*(M)$ generated by quaternions. However, for any $SU(2)$-invariant form Θ, and any Hermitian metric g, one has $\Theta \wedge \Theta = -\|\Theta\|_g^2 \operatorname{Vol}_g$. On the other hand, Θ is exact, because the canonical bundle of a K3 is trivial. **This implies that the Obata connection on the canonical bundle $K_{M,I}$ is flat**. Given that $\pi_1(K3) = 0$, **we obtain that $K_{M,I}$ is trivialized by an Obata-parallel section.**

Step 2: The Obata-parallel sections of the canonical bundle are closed 2-forms (any parallel differential form is closed, if the connection is torsion-free). Varying the complex structure, **we obtain a rank 3 space W of parallel differential forms,** $\omega_I, \omega_J, \omega_K$; the corresponding metric is hyperkähler, because its holonomy is in Sp(1). ■

Twistor spaces for hypercomplex manifolds

DEFINITION: Induced complex structures on a hypercomplex manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\mathrm{Tw}(M)$ of a hypercomplex manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\mathrm{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x \mathrm{Tw}(M) \to T_x \mathrm{Tw}(M)$ satisfies $I_{\mathrm{Tw}}^2 = -\mathrm{Id}$. **It defines an almost complex structure on $\mathrm{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, then $\mathrm{Tw}(M) = \mathrm{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\mathrm{Tw}(M)$ never admits a Kähler structure.

Rational curves on $\operatorname{Tw}(M)$.

DEFINITION: An ample rational curve on a complex manifold M is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$. It is called **a quasiline** if all $i_k = 1$.

THEOREM: ("twistor spaces are rationally connected")
Let M be a compact complex manifold containing a an ample rational line. **red any N points $z_1, ..., z_N$ can be connected by an ample rational curve.**

CLAIM: Let M be a hyperkähler manifold, $\operatorname{Tw}(M) \xrightarrow{\sigma} M$ its twistor space, $m \in M$ a point, and $S_m = \mathbb{C}P^1 \times \{m\}$ the corresponding rational curve in $\operatorname{Tw}(M)$. **Then S_m is a quasiline.**

Proof: Since the claim is essentially infinitesimal, it suffices to check it when M is flat. **Then $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus 2p}) \cong \mathbb{C}P^{2p+1} \backslash \mathbb{C}P^{2p-1}$, and S_m is a section of $\mathcal{O}(1)^{\oplus 2p}$.** ∎

The twistor data

Let $\check\tau$ denote the central symmetry on $\mathbb{C}P^1$; if we identify $\mathbb{C}P^1$ with imaginary unit quaternions, we have $\check\tau(L) = -L$. It is **an anticomplex involution without fixed points.**

DEFINITION: The **twistor data** is a complex manifold Tw equipped with the following structures.

1. **A holomorphic submersion** $\pi :$ Tw $\longrightarrow \mathbb{C}P^1$ **and an anticomplex involution** $\tau :$ Tw $\longrightarrow$ Tw which makes this diagram commutative

$$\begin{array}{ccc} \mathrm{Tw} & \stackrel{\tau}{\longrightarrow} & \mathrm{Tw} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{C}P^1 & \stackrel{\check\tau}{\longrightarrow} & \mathbb{C}P^1 \end{array}$$

2. **A connected component** Hor **in the set** $\mathrm{Sec}^\tau \subset \mathrm{Sec}$ **of** τ**-invariant sections of** π such that for each $S \in$ Hor, the normal bundle to S is $\mathcal{O}(1)^{2n}$ and for each point $x \in$ Tw there exists a unique $S \in$ Hor passing through x.

REMARK: With any twistor space Tw(M) of a hypercomplex manifold, **one associates the twistor data in a natural way:** $\tau(I, m) = (-I, m)$, and Hor(M) the space of all sections S_m taking $I \in \mathbb{C}P^1$ to $(I, m) \in$ Tw(M), where $m \in M$ is a fixed point.

Hypercomplex structures defined in terms of twistor data

THEOREM: (HKLR)
Let M be a hypercomplex manifold. Then **the twistor data on** $\operatorname{Tw}(M)$ **can be used to recover the hypercomplex structure on** M**, which is identified with** Hor**.** Moreoved, **for any twistor data** $(\operatorname{Tw}, \tau, \operatorname{Hor})$**, there exists a hypercomplex structure** (I, J, K) **on** Hor **such that these twistor data are associated with** (I, J, K)**.**

Proof: *N. J. Hitchin, A. Karlhede, U. Lindström, M. Roček,* Hyperkähler metrics and supersymmetry, *Comm. Math. Phys.* **108** *(1987), 535-589.* ∎

Complex tori

DEFINITION: A complex torus is a complex manifold M such that its Albanese map Alb : $M \longrightarrow \frac{H^0(\Omega^1 M)^*}{H^1(M,\mathbb{Z})}$ is an isomorphism.

REMARK: Any Kähler-type complex structure on a manifold diffeomorphic to a torus has this nature; there are **non-Kähler complex structures on a torus,** not well understood yet. These complex structures don't give "complex torus", because the Albanese map for such manifolds is never an isomorphism.

THEOREM: (F. Catanese)
Let $\mathcal{X}$ be a connected, continuous family of complex structures on a manifold M diffeomorphic to a torus. Assume that for some $I \in \mathcal{X}$, the manifold (M, I) is a complex storus. **Then (M, I_1) is a torus for all $I' \in \mathcal{X}$.**

Proof: Fabrizio M.E. Catanese, *Deformation types of real and complex manifolds*, arXiv:math/0111245, Theorem 4.1. ■

Translations and flat structures on complex tori

REMARK: Let $\theta_1, ...\theta_n$ be holomorphic differentials on a complex torus M. Their antiderivatives define a flat affine chart on M; the corresponding flat affine structure on M is canonically defined. **This also defines a holomorphic flat affine connection on M.**

REMARK: Also, each complex torus M is a torsor over the corresponding group manifold, identified with a connected component $\operatorname{Aut}_0(M)$ of $\operatorname{Aut}(M)$, and its action on M is canonically defined. Since $\operatorname{Aut}_0(M)$ is (non-canonically) identified with M, this action is called **the action of the torus on itself by translations**.

Exotic holomorphic structures on a torus are flat

Theorem 1: Let (I, J, K) be a hypercomplex structure on a complex torus (M, I), and ∇ its Obata connection. **Then ∇ is flat.**

Proof. Step 1: Any anticomplex involution of a torus exchanges holomorphic and antiholomorphic differentials, hence preserves the flat structure. Since the fibers of $\pi: \operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$ are flat, the universal covering $\tilde{\operatorname{Tw}}(M)$ is an affine bundle, and the anticomplex involution preserves the affine structure. Fixing a horizontal section, **we identify $\tilde{\operatorname{Tw}}(M)$ with $\operatorname{Tot}(\mathcal{O}(1)^{2n})$; the anticomplex involution also preserves the vector bundle structure.**

Step 2: Since the hypercomplex structure on $\operatorname{Tot}(\mathcal{O}(1)^{2n}) = \tilde{\operatorname{Tw}}(M)$ is linear, it gives a hypercomplex structure, compatible with the vector bundle operation (addition and multiplication). Such a hypercomplex structure is flat. **We obtain that (M, I, J, K) is a quotient of a flat hypercomplex manifold $\mathbb{H}^n$ by an affine action of $\mathbb{Z}^{4n}$.** ■

REMARK: If the holonomy of Obata connection on M is trivial (or just compact), it would immediately follow that M is a hyperkähler torus. However, **this is false,** even for a torus obtained as a compact quotient of $\mathbb{H}^n$ by $\mathbb{Z}^{4n}$.

Flat affine structures and the development map

DEFINITION: A **flat affine structure** on a manifold M is a flat torsion-free connection.

DEFINITION: Let M be a simply connected flat affine manifold, and $\theta_1, ..., \theta_n \in \Lambda^1 M$ a basis of parallel 1-forms. Since a parallel 1-form is closed and $H^1(M, \mathbb{R}) = 0$, the forms θ_i are exact. Then $\theta_i = dx_i$. The map $\delta : M \to \mathbb{R}^n$ taking m to $(x_1(m), ..., x_n(m))$ is called **the development map.** We consider $\mathbb{R}^n$ as a flat affine manifold, with the standard flat affine structure.

CLAIM: The development map $\delta : M \to \mathbb{R}^n$ **is compatible with the flat affine connections.**

Proof: It takes the coordinate 1-forms $dx_1, ..., dx_n \in \Lambda^1(M)$ to $\theta_1, ..., \theta_n \in \Lambda^1 M$. However, these 1-forms are parallel. ■

Linear and affine holonomy

DEFINITION: **Linear holonomy** (or **holonomy**) of a flat affine connection ∇ is its monodromy in TM; by definition, the holonomy group belongs to $GL(T_xM)$, where $x \in M$ is a base point.

DEFINITION: Let Aff$(\mathbb{R}^n)$ denote the group of affine transforms of $\mathbb{R}^n$. Clearly, Aff$(\mathbb{R}^n)$ is a semidirect product, Aff$(\mathbb{R}^n) = GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$. The natural map Aff$(\mathbb{R}^n) \longrightarrow GL(n,\mathbb{R})$ is called **the linearization.**

DEFINITION: Let M be a flat affine n-manifold, Aff$(\mathbb{R}^n)$ $\tilde{M}$ its universal cover $\delta : \tilde{M} \to \mathbb{R}^n$ the development map, and $a : \pi_1(M) \longrightarrow$ Aff$(\mathbb{R}^n)$ the map taking $\gamma \in \pi_1(M)$ to an element of Aff$(\mathbb{R}^n)$ making the following diagram commutative:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\delta} & \mathbb{R}^n \\ \gamma \Big\downarrow & & \Big\downarrow a \\ \tilde{M} & \xrightarrow{\delta} & \mathbb{R}^n. \end{array}$$

The map $a : \pi_1(M) \longrightarrow$ Aff$(\mathbb{R}^n)$ is called **the affine holonomy map.**

REMARK: The linear holonomy of a manifold **is the linearization if its affine holonomy.**

Non-standard flat affine structures on a torus

REMARK: A flat affine structure on a torus is called **standard** if its linear holonomy is trivial.

Remark 1: Let (M, ∇) be a flat affine torus with the standard flat affine structure. Then $\pi_1(M)$ acts on $\tilde{M}$ by translations, hence $\tilde{M} = \mathbb{R}^n$ **and** M **is isomorphic to** $\mathbb{R}^n/\mathbb{Z}^n$ **with the standard flat affine structure.**

REMARK: In *Sullivan, Dennis; Thurston, William Manifolds with canonical coordinate charts: some examples. Enseign. Math. (2) 29 (1983), no. 1-2, 15-25.* Thurston and Sullivan gave examples of non-standard flat affine structures on a torus.

EXAMPLE: Consider the quotient $M := \frac{\mathbb{R}^2 \backslash 0}{\mathbb{Z}}$, where $\mathbb{Z}$ acts by homotheties. **Clearly, the holonomy of** M **is** $\mathbb{Z}$ **acting on** TM **by homotheties.**

Example 1: Consider $\mathbb{Z}^2$-action ρ on $\mathbb{R}^2$ generated by $(x, y) \to (x + 1, y)$ and $(x, y) \to (x + y, y + 1)$. The projection to the second component maps $\frac{\mathbb{R}^2}{\operatorname{im} \rho}$ to S^1, with the fiber S^1, hence $\frac{\mathbb{R}^2}{\operatorname{im} \rho}$ is a torus; its (linear) holonomy is generated by $A(x, y) := (x + y, y)$.

Exotic hypercomplex structures on a torus: examples

Corollary 1: Let (M,I,J,K) be a hypercomplex manifold, and ∇ its Obata connection. Assume that (M,I) is a compact complex torus. **Then ∇ is flat, and the hypercomplex structure is exotic if and only if the (linear) holonomy of ∇ is non-trivial.**

Proof: The connection ∇ is flat by Theorem 1. If its holonomy is trivial, M is a quotient of $\mathbb{H}^n$ by translations, as follows from Remark 1. ■

EXAMPLE: Let $e_1, ..., e_4 \in \mathbb{H}$ be a basis in quaternions. Consider the following action of $\mathbb{Z}^8 = \langle t_1, ..., t_8 \rangle$ on $\mathbb{H}^2$: for $i = 1, ..., 4$, we have $t_i(h,h') = (h + e_i, h')$ for $i = 5, 6, 7, 8$, we have $t_i(h,h') = (h + h', h' + e_{i-4})$. The quotient $\frac{\mathbb{H}^2}{\mathbb{Z}^8}$ is diffeomorphic to an 8-torus by the same reason as in Example 1. The action of $\mathbb{Z}^8$ on $\mathbb{H}^2$ is $\mathbb{H}$-linear, hence the quotient is hypercomplex, with Obata connection ∇ induced by the flat connection on $\mathbb{H}^2$. However, **the linear holonomy of ∇ contains the map $A(h,h') := (h + h', h)$, hence it is non-standard and the hypercomplex structure is exotic** (Corollary 1).

Frid-Goldman-Hirsch theorem

DEFINITION: A flat affine manifold (M, ∇) is called **complete** if $M = \frac{\mathbb{R}^n}{\Gamma}$, where $\Gamma = \pi_1(M)$, with its action factorized through $\mathsf{Aff}(\mathbb{R}^n)$.

CONJECTURE: ("Marcus conjecture") A compact flat affine manifold **is complete if and only if it admits a parallel volume form.**

THEOREM: Let (M, ∇) be a compact flat affine manifold with affine holonomy group nilpotent. **Then the following are equivalent:**

(a) (M, ∇) is complete,

(b) (M, ∇) admits a paralell volume form, and

(c) its linear holonomy action is unipotent.

Proof: Theorem A in *Fried, D., Goldman, W., Hirsch, M.W., Affine manifolds with nilpotent holonomy, Commentarii Mathematici Helvetici 56, 487-523 (1981),* `https://doi.org/10.1007/BF02566225` ∎

Frid-Goldman-Hirsch theorem for exotic hypercomplex structures

COROLLARY: Let $W := \mathbb{H}^n$, (M, I, J, K) an exotic hypercomplex structure on a torus, and ∇ its Obata connection. **Then (M, ∇) satisfies (a)-(c) of Frid-Goldman-Hirsch theorem.**

Proof: Since (M, I) is Kähler, it is HKT; since its canonical bundle is trivial and (M, I, J, K) is HKT, the Obata holonomy is contained in $SL(n, \mathbb{H})$ and ∇ fixes a volume form, as shown in *M. Verbitsky, Hyperkähler manifolds with torsion, supersymmetry and Hodge theory, Asian J. of Math., Vol. 6 (4), December 2002).* ∎

Category of flat affine tori

Consider **the category of flat affine tori,** with the morphisms smooth maps $X \longrightarrow Y$ compatible with the flat affine connection (that is, mapping parallel forms, local in Y, to parallel forms on X).

DEFINITION: An exact sequence of flat affine tori is a sequence

$$0 \longrightarrow M_1 \xrightarrow{a} M_2 \xrightarrow{b} M_3 \longrightarrow 0$$

where all M_i are flat affine tori, all maps are morphisms, the map b is submersive, and a injectively mapping M_1 to a fiber of b.

REMARK: Exact sequences of flat affine tori **correspond to exact sequences of $\mathbb{Z}^n$-action on $\mathbb{R}^n$ factorizing through** $\mathsf{Aff}(\mathbb{R}^n)$**.**

Integer lattice preserved by the holonomy

PROPOSITION: Let (M,∇) be a flat affine connection on a torus $M=\frac{W}{\mathbb{Z}^n}$, satisfying the Frid-Goldman-Hirsch conditions (a)-(c). Denote by W_{lin} the linearization of W, and let $\rho:\Gamma\longrightarrow GL(W_{\text{lin}})$ be the linear holonomy. **Then $\rho(\Gamma)$ preserves a cocompact integer lattice $\Lambda\subset W_{\text{lin}}$.**

Proof: Since ρ is unipotent, there exists a filtration $0=W_0\subset ...\subset W_k=W_{\text{lin}}$ preserved by ρ which acts trivially on each subquotient W_i/W_{i-1}. Then Γ acts on W_{lin}/W_{k-1} by parallel transport, which defines a morphism of flat affine manifolds $(M,\nabla)=\longrightarrow\frac{W_{\text{lin}}/W_{k-1}}{\Lambda_k}$, where Λ_k is a cocompact lattice. This gives an exact sequence of flat affine tori $0\longrightarrow M'\longrightarrow M\longrightarrow\frac{W_{\text{lin}}/W_{k-1}}{\Lambda_k}$. Using induction in $\dim M$, we may assume that $M'=\frac{W'}{\mathbb{Z}^{n'}}$, with W'_{lin} admitting a $\rho(\mathbb{Z}^{n'})$-invariant lattice. The leftmost and rightmost terms of the exact sequence $0\longrightarrow W'_{\text{lin}}\longrightarrow W_{\text{lin}}\longrightarrow W_{\text{lin}}/W_{k-1}\longrightarrow 0$ are equipped with a holonomy-invariant lattice, hence W_{lin} also admits a holonomy-invariant lattice. ■

COROLLARY: Let (M,I,J,K) be an exotic hypercomplex structure on a torus, and ∇ its Obata connection. **Then (M,∇) is a flat affine torus admitting an exact sequence** $0\longrightarrow M'\longrightarrow M\longrightarrow T\longrightarrow 0$, where M' is a hypercomplex flat affine torus, and T is a hypercomplex (and, therefore, hyperkähler) torus with trivial linear holonomy. ■

Twistor space of an exotic hypercomplex torus

THEOREM: Let (M,I,J,K) be an exotic hypercomplex structure on a compact complex torus, and $\operatorname{Tw}(M) \stackrel{\pi}{\longrightarrow} \mathbb{C}P^1$ its twistor space. **Then $\operatorname{Tw}(M)$ is isomorphic to the twistor space of a hyperkähler torus.**

Proof. Step 1: The twistor projection $\operatorname{Tw}(M) \stackrel{\pi}{\longrightarrow} \mathbb{C}P^1$ is a smooth holomorphic fibration, its fibers are complex tori. Consider the variation of Hodge structures over $\mathbb{C}P^1$ associated with the first cohomology of the fibers of $\operatorname{Tw}(M)$. Since any torus bundle possessing a section is determined by its variation of Hodge structures, **it suffices to show that this variation of Hodge structures is isomorphic to one associated with a hyperkäler structure on a torus.**

Step 2: Let $s: \mathbb{C}P^1 \longrightarrow \operatorname{Tw}(M)$ be a horizontal section associated with $m \in M$. Then the normal bundle Ns of s is $\mathcal{O}(1)^{2n}$ (this is always true for twistor spaces of hypercomplex manifolds). For any $I \in \mathbb{C}P^1$, we have $H^{1,0}(\pi^{-1}(I)) = (Ns\big|_I)^*$, because $\Omega^1(\pi^{-1}(I))$ is a trivial vector bundle on the torus $\pi^{-1}(I)$. This identifies the bundle $R^1\pi_*(\mathbb{C})$ with $Ns \otimes_{\mathbb{R}} \mathbb{C}$. This bundle is trivial with the fiber $T_mM \otimes_{\mathbb{R}} \mathbb{C}$, and its Hodge decomposition in $I' \in \mathbb{C}P^1$

is determined by the action of the quaternion I' on T_mM. **This implies that the variation of Hodge structures on $R^1\pi_*(\mathbb{C})$ is determined by the quaternionic structure on T_mM**, hence this variation of Hodge structure coincides with one obtained from $(T_mM/\mathbb{Z}^{4n}, I, J, K)$. ■

REMARK: The exotic hypercomplex structure can be recovered from the twistor data: the twistor space, anticomplex involution and a component in the space of real sections. The twistor space itself is standard as we have just shown. The space of twistor section is identified with $H^0(\mathcal{O}(1)^{2n})$ by homotopy lifting lemma. Therefore, **the exotic properties of the hypercomplex structure are born by the anticomplex involution on its twistor space.**