# Any component of moduli of polarized hyperkahler manifolds is dense in its deformation space

Misha Verbitsky

**Moduli spaces and automorphic forms**

**CIRM, Luminy, 11.10.11**

## Plan of the talk

**This is a joint work with Sasha Anan′in (UNICAMP)**

1. Introduce hyperkähler manifolds and their moduli. Define **the birational moduli space as a quotient of a Teichmüller space** $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ **by an arithmetic group** $\Gamma_I$**.**

2. Define the polarized Teichmüller space as a divisor in $\mathbb{P}\mathrm{er}$.

3. Show that **its image in the moduli space** $\mathbb{P}er/\Gamma_I$ **is dense.**

4. Explain the background story: why is this interesting.

## Holomorphically symplectic manifolds

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**DEFINITION:** For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

**DEFINITION:** A hyperkähler manifold $M$ is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov's decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Further on, all hyperkähler manifolds are assumed to be simple.**

## The Teichmüller space and the mapping class group

**Definition:** Let $M$ be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on $M$, and let $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

**Remark:** Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

**Definition:** Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of $M$. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on $M$** is a connected component of $\operatorname{Teich}/\Gamma$.

**Remark:** This terminology is **standard for curves.**

**REMARK:** For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

## The Bogomolov-Beauville-Fujiki form

**THEOREM:** (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where $M$ is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form $q$ on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

**Definition:** This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = & \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ & -\frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)\end{aligned}$$

where $\Omega$ is the holomorphic symplectic form, and $\lambda > 0$.

**Remark:** **$q$ has signature $(b_2 - 3, 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where $\omega$ is a Kähler form.

## Computation of the mapping class group

**Theorem:** (Sullivan) Let $M$ be a compact, simply connected Kähler manifold, $\dim_{\mathbb C} M \geqslant 3$. Denote by $\Gamma_0$ the group of automorphisms of an algebra $H^*(M,\mathbb Z)$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in $\Gamma_0$.**

**Theorem:** Let $M$ be a simple hyperkähler manifold, and $\Gamma_0$ as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb Z)}$ **is a finite index subgroup of $O(H^2(M,\mathbb Z),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb Z),q)$ **has finite kernel.**

## The period map

**Remark:** For any $J \in$ Teich, $(M,J)$ is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

**Definition:** Let $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map $J$ to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

**REMARK:** $P$ **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of $M$.

**REMARK:** $\mathbb{P}er = SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$

## Birational Teichmüller moduli space

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**DEFINITION:** The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of $M$.

**THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism,** for each connected component of $\operatorname{Teich}_b$.

**DEFINITION:** Let $M$ be a hyperkaehler manifold, $\operatorname{Teich}_b$ its birational Teichmüller space, and $\Gamma$ the mapping class group. The quotient $\operatorname{Teich}_b/\Gamma$ is called **the birational moduli space** of $M$.

**THEOREM:** Let $(M, I)$ be a hyperkähler manifold, and $W$ a connected component of its birational moduli space. **Then $W$ is isomorphic to $\mathbb{P}\mathrm{er}/\Gamma_I$, where $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and $\Gamma_I$ is an arithmetic group in $O(H^2(M, \mathbb{R}), q)$.**

**A CAUTION:** Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

## Polarized Teichmüller space

**THEOREM:** (Demailly, Paun, Huybrechts, Boucksom) Let $M$ be a hyperkähler manifold, such that all integer (1,1)-classes satisfy $q(\nu,\nu) \geqslant 0$. **Then its Kähler cone is one of two connected components of the set** $K := \left\{\nu \in H^{1,1}(M,\mathbb{R}) \;\middle|\; q(\nu,\nu) > 0\right\}$**.**

**REMARK:** $H^{1,1}(M,\mathbb{R})$ is an orthogonal complement to $\langle\Omega,\overline{\Omega}\rangle$, where $\Omega$ is the cohomology class of holomorphic symplectic form.

**DEFINITION:** Define $\operatorname{Teich}_\eta \subset \operatorname{Teich}$ to be the set of all $I$ with $l := \operatorname{Per}(I) \in \mathbb{P}\mathrm{er} \subset \mathbb{P}H^2(M,\mathbb{C})$ satisfying $l \in \langle\Omega,\overline{\Omega}\rangle^\perp$.

**COROLLARY:** Let $\eta \in H^2(M,\mathbb{Z})$ be a vector which satisfies $q(\eta,\eta) > 0$. Then, **for a general** $I \in \operatorname{Teich}_\eta$**, the class** $\eta$ **is ample on** $(M,I)$**.**

**DEFINITION:** We call $\operatorname{Teich}_\eta$ **the polarized Teichmüller space** of $M$. It is a divisor in Teich.

**REMARK:** $\eta$ is not necessarily ample in **non-generic** points of $\operatorname{Teich}_\eta$. However, the set of $I$ for which $\eta$ is ample is **open and dense in** $\operatorname{Teich}_\eta$.

**Polarized moduli space**

**DEFINITION:** Let $\Gamma_{I,\eta}$ be a subgroup of a mapping class group preserving $\eta$ and a given component of $\mathrm{Teich}_\eta$. Define **the polarized birational moduli space** as a quotient $(\mathrm{Teich}_\eta/\sim)/\Gamma_{I,\eta}$.

**THEOREM: (Anan'in-V.; this is the main result of this talk)**
For each $\eta$, **the birational polarized moduli space is dense** in the birational moduli space $\mathrm{Teich}_b/\Gamma$.

**REMARK:** For a K3 surface, **the moduli of quartic surfaces is dense in the moduli of all K3 surfaces.** The original proofs of many results on K3 surfaces were based on this observation.

## Polarized moduli space and the 2-plane Grassmannian

The density theorem is implied by the following algebraic observation.

**THEOREM:** Let $V_{\mathbb{Z}} \subset V$ be an integer lattice in a vector spac $V$ equipped with an integral bilinear form of signature $(+++,-...-)$. Given a positive integer vector $\eta \in V_{\mathbb{Z}}$, denote by $\mathrm{Gr}^{++}(\eta^\perp) \subset \mathrm{Gr}^{++}(V)$ the space of all oriented, positive 2-planes orthogonal to $\eta$. Consider a finite index subgroup $G \subset SO(V_{\mathbb{Z}})$ acting on $\mathrm{Gr}^{++}(V)$ in a natural way. **Then $G \cdot \mathrm{Gr}^{++}(\eta^\perp)$ is dense in $\mathrm{Gr}^{++}(V)$.**

For our purposes, $V_{\mathbb{Z}} = (H^2(M,\mathbb{Z}), q)$ is a lattice of second cohomology of a hyperkähler manifold, and $\mathrm{Gr}^{++}(\eta^\perp) = \mathrm{Per}(\mathrm{Teich}_\eta)$, and $G = \Gamma_I$ the subgroup of the mapping class group preserving a connected component $\mathrm{Teich}_I$.

Then, **density of $G \cdot \mathrm{Gr}^{++}(\eta^\perp)$ in $\mathrm{Gr}^{++}(V)$ is equivalent to the density of the image of $\mathrm{Teich}_\eta$ in the birational moduli space $\mathrm{Teich}_b/\Gamma_I$.**

**REMARK:** When $V_{\mathbb{Z}}$ is unimodular, the group $SO(V_{\mathbb{Z}})$ **acts transitively on the set of integer vectors of a given length in $V_{\mathbb{Z}}$** ("Eichler's criterion"). Therefore, the orbit $G \cdot \eta$ is dense in $\mathbb{P}(V)$, hence $G \cdot \mathrm{Gr}^{++}(\eta^\perp)$ is dense in $\mathrm{Gr}^{++}(V)$.

**When $V_{\mathbb{Z}}$ is not unimodular, this is false.**

## Proof of the density theorem

**Step 1: It suffices to prove it for $V$ of signature $(+++-)$.** Since the rational subspaces are dense in $\mathrm{Gr}_{++}(V)$, it suffices to show that any rational 2-plane $C \in \mathrm{Gr}_{++}(V)$ belongs to the closure of $G{\cdot}\mathrm{Gr}_{++}(\eta^\perp)$. Choose a rational space $V_0$ of signature $(3,1)$ containing $C$ and $\eta$, and notice that $G \cap SO(V_0)$ has finite index in $SO(V_0, \mathbb{Z})$.

**From now on we assume that $V$ has signature $(3,1)$.**

**Step 2:** Using the homeomorphism $\mathrm{Gr}_{++}(V) \to \mathrm{Gr}_{+-}(V)$, $G \mapsto G^\perp$ mapping the subspaces of signature $++$ to their orthogonal complements (of signature $+-$, we reformulate the density theorem as follows: **Every rational plane $C_0 \in \mathrm{Gr}_{+-}(V)$ belongs to the closure of $G \cdot \left\{R \in \mathrm{Gr}_{+-}(V) \;\middle|\; R \ni l\right\}$.**

**Step 3:** Consider the space $W$ (of signature $++-$) generated by $R$ and $l$. **The statement of Step 3 is implied by the following lemma.**

**LEMMA:** Let $V$ be an $\mathbb{R}$-vector space equipped with a symmetric form of signature $++-$, $G$ a subgroup of finite index in $\varnothing(V_\mathbb{Z})$, where $V_\mathbb{Z}$ is a lattice in $V$, and $l \in V$ a positive vector. **Then $G \cdot \left\{C \in \mathrm{Gr}_{+-}(V) \;\middle|\; C \ni l\right\}$ is dense in $\mathrm{Gr}_{+-}(V)$.**

## The density lemma and the hyperbolic plane

Now we prove the lemma stated above:

**LEMMA:** Let $V$ be an $\mathbb{R}$-vector space equipped with a symmetric form of signature $++-$, $G$ a subgroup of finite index in $\varnothing(V_{\mathbb{Z}})$, where $V_{\mathbb{Z}}$ is a lattice in $V$, and $l \in V$ a positive vector. **Then $G \cdot \left\{C \in \mathrm{Gr}_{+-}(V) \;\middle|\; C \ni l\right\}$ is dense in $\mathrm{Gr}_{+-}(V)$.**

It is a statement about geometry of the hyperbolic plane $\mathcal{H}$.

**REMARK:** Identify $\mathcal{H}$ with the projectivization of the negative cone in $V$. **Any geodesic on $\mathcal{H}$ is obtained as a projectivization of a subspace $C \in \mathrm{Gr}_{+-}(V)$.**

**CLAIM:** Let $l$ be a positive vector in $V$, and $C$ a $(+-)$-plane. **Then $l \in C$ iff the geodesic $\mathbb{P}C$ is orthogonal to the geodesic $\mathbb{P}l^\perp \in \mathrm{Gr}_{+-}(V)$.**

We reduced the lemma above to the following claim.

**CLAIM:** Let $V$ be a vector space of signature $++-$, $C, C'$ a pair of distinct geodesics, and $G$ a subgroup of finite index in $SO(V_{\mathbb{Z}})$. Then for some $\gamma \in G$, $C$ intersects $\gamma(C')$, and **the set of angles $\{\alpha = \angle(C, \gamma(C')) \;\;|\;\; \gamma \in G\}$ between those geodesics is dense in $[0, \pi]$.**

## Open questions

**QUESTION:** This proof does not work when $q(\eta, \eta) \leqslant 0$. **Is the density theorem true for such $\eta$?** It should be.

**QUESTION:** Let $M_\eta$ be the polarized moduli space. It is quasiprojective and admits a complex analytic compactification (Baily-Borel compactification) **Is there a way to compactify the moduli space of all hyperkähler manifolds?** Should follow from Gromov's compactification.

**QUESTION:** Tosatti has shown that the **Gromov's limits have algebraic properties as long as the volume of the Kähler form stays bounded from below.** In particular, it is compatible with the Viehweg's universal family on the Baily-Borel compactification (proven in 1980-ies by Kobayashi, Todorov for K3). Is there a way to relate Gromov's limits of the set of **all** Ricci-flat metrics on $M$ with algebraic geometry?

**REMARK:** Gromov's collapse has **obvious applications to the SYZ/abundance conjecture for hyperkähler manifolds.**