Symplectic geometry

lecture 14: Kähler reduction

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Characteristic foliation (reminder)

DEFINITION: Let (M,ω) be a symplectic manifold. A submanifold $Z \subset M$ is called **coisotropic** if $\dim_{\mathbb{R}} Z \geqslant \dim_{\mathbb{R}} M$ and $\omega|_Z$ has rank $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} Z$ (minimal possible), or, equivalently, $(TZ)^{\perp_\omega} \subset TZ$, where $(TZ)^{\perp_\omega} := \{x \in TZ \mid i_x\omega = 0\}$

DEFINITION: Let $Z \subset (M,\omega)$ be a coisotropic submanifold. The bundle $(TZ)^{\perp_\omega}$ is called **the characteristic bundle of Z**.

THEOREM: Let $Z \subset (M,\omega)$ be a coisotropic submanifold, and $K \subset TM$ its characteristic bundle. **Then $[K,K] \subset K$, hence K is tangent to a foliation** $\mathcal{F}$ which is called **the characteristic foliation of Z.** Moreover, **the restriction $\omega|_Z$ is basic, and symplectic on the leaf space of the characteristic foliation.**

Moment maps (reminder)

DEFINITION: (M, ω) be a symplectic manifold, and G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**, that is, up to addition of a central vector $c \in \mathfrak{g}^*$.

CLAIM: An equivariant moment map exists whenever $H^1(G, \mathfrak{g}^*) = 0$. In particular, when G is reductive and M is simply connected, an equivariant moment map exists. Further on, all moment maps will be tacitly considered equivariant.

Weinstein-Marsden theorem (reminder)

DEFINITION: (Weinstein-Marsden) (M, ω) be a symplectic manifold, G a compact Lie group freely acting on M by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of M, denoted by $M/\!\!/G$.

CLAIM: The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

Proof. Step 1: $T_x(\mu^{-1}(c)) = d\mu^{-1}(0)$. However, the space $\langle d\mu, g\rangle \subset \Lambda^1 M$ is ω-dual to the space $\tau(\mathfrak{g})$ of vector fields tangent to the G-action, **hence $d\mu^{-1}(c) = \tau(\mathfrak{g})^\perp$.**

Step 2: Since μ is G-equivariant, G preserves $\mu^{-1}(c)$, hence $\tau(\mathfrak{g}) \subset d\mu^{-1}(0)$. This implies that **$\tau(\mathfrak{g}) \subset TM$ is isotropic** (that is, $\omega\big|_{\tau(\mathfrak{g})} = 0$). Its ω-orthogonal complement in T_xM is $T_x(\mu^{-1}(c))$ (Step 1).

Step 3: Consider the **characteristic foliation** $\mathcal{F}$ on $\mu^{-1}(c)$. It is a bundle because $\mu^{-1}(c) \subset M$ is coisotropic. From Step 2 **we obtain that $\mathcal{F} = \tau(\mathfrak{g})$.**

Step 4: Since $\omega\big|_{\mu^{-1}(c)}$ is closed, it satisfies $\operatorname{Lie}_v(\omega) = 0$ for all $v \in \mathcal{F}$. This implies that it is *basic*, that is, lifted from the leaf space of characteristic foliation, identified with $M/\!\!/G$. ■

Kähler manifolds

DEFINITION: Kähler manifold is a complex manifold equipped with a compatible symplectic structure.

EXAMPLE: A complex submanifold $Z \subset (M, I, \omega)$ of a Kähler manifold is also Kähler. Indeed, a restriction of a symplectic form to Z is closed; it is non-degenerate because $\omega(x, Ix)$ is positive definite, hence ω is non-degenerate on every I-invariant subspace.

EXAMPLE: The Fubini-Studi form ω on $\mathbb{C}P^n$ is Kähler.

DEFINITION: A complex submanifold of $\mathbb{C}P^n$ is called **projective**.

EXAMPLE: All projective manifolds are Kähler.

EXAMPLE: Compact complex tori **are always Kähler, but not always projective.**

Symplectic reduction and a Kähler potential

DEFINITION: Let $d^c := IdI^{-1}$. **Kähler potential** on a Kähler manifold (M,ω) is a function ψ such that $dd^c\psi = \omega$.

PROPOSITION: Let G be a real Lie group acting on a Kähler manifold M by holomorphic isometries, and ψ a G-invariant Kähler potential. **Then the moment map $\mathfrak{g} \times M \xrightarrow{\mu_g} \mathbb{R}$ can be written as $g, m \longrightarrow \operatorname{Lie}_{Iv}\psi$,** where $v = \tau(g) \in TM$ is the tangent vector field associated with $g \in \mathfrak{g}$.

Proof: Since ψ is G-invariant, and I is G-invariant, we have $0 = \operatorname{Lie}_v d^c\psi = i_v(dd^c\psi) + d(i_v d^c\psi)$. Using $\omega = dd^c\psi$, we rewrite this equation as $i_v\omega = -d(\langle d^c\psi, v\rangle)$, giving an equation for the moment map $\mu_g = -\langle d^c\psi, v\rangle$. Acting by I on both sides, we obtain $\mu_g = \langle d\psi, Iv\rangle = \operatorname{Lie}_{Iv}\psi$. ■

COROLLARY: Let V be a Hermitian representation of a compact Lie group G. **Then the corresponding moment map can be written as $\mu_g(v) = \operatorname{Lie}_{Ig}|v|^2 = \frac{1}{4}\langle v, Ig(v)\rangle$.** ■

Transversal complex and Riemannian structures

DEFINITION: A foliation on a manifold M is a sub-bundle $B \subset TM$ such that $[B,B] \subset B$. By Frobenius theorem, this is equivalent to a local decomposition $U = S \times R$, of any sufficiently small open set $U \subset M$, with $B \subset TU$ equal to the tangent bundle to the fibers of the projection $U \longrightarrow R$. **A leaf** of a foliation is a maximal connected immersed submanifold $Z \longrightarrow M$ which satisfies $T_zZ = B|_z$ at each $z \in Z$. **Projection to the leaf space** is a smooth submersion $U \longrightarrow R$, mapping U to the set of leaves of B on U.

DEFINITION: Transversal Riemannian structure / symplectic structure / almost complex structure on a foliated manifold $(M, B \subset TM)$ is a scalar product / skew-symmetric form/almost complex structure on the bundle (TM/B) which is locally obtained as a pullback of a Riemannian /symplectic / almost complex structure on the leaf space.

Transversal complex and Riemannian structures (2)

PROPOSITION: Let M be a manifold equipped with a locally free action of a compact Lie group G and $B \subset TM$ the bundle of vectors tangent to the G-action. Suppose that TM/B is equipped with a G-invariant metric h and a G-invariant almost complex structure I. **Then these structures are transversal.**

Proof. Step 1: Let $h \in \operatorname{Sym}^2(T^*M)$ be a symmetric 2-form vanishing on B and positive definite and G-invariant on TM/B. In Lecture 13 we proved that **a B-basic differential form α is the one which satisfies $i_X\alpha = 0$ and $\operatorname{Lie}_X \alpha = 0$ for all vector fields $X \in B$.** The same argument can be applied to show that h is basic whenever $\operatorname{Lie}_X h = 0$ for all $X \in B$. This follows because h is G-invariant.

Step 2: It remains to show that the 2-form $\omega \in \Lambda^2 M$ obtained as $\omega(x,y) = h(x, Iy)$ is basic. This form vanishes on B and is G-invariant, hence it is basic by the same argument. ■

Symplectic reduction for almost Kähler manifolds

THEOREM 1: Let M be an almost Kähler manifold, G a compact Lie group acting on M freely by Hamiltonian isometries, $t \in \mathfrak{g}^*$ a central element, and $\mu: M \longrightarrow \mathfrak{g}^*$ an equivariant moment map. Denote by $K \subset T\mu^{-1}(t)$ the bundle of vectors tangent to the orbits of G on $\mu^{-1}(t)$. Then $K \subset T\mu^{-1}(t)$ is tangent to a foliation, **equipped with a transversal Riemannian and a transversal symplectic structure,** obtained by restricting the Riemannian and symplectic forms to $K^\perp$. Moreover, these bilinear forms **define an almost Kähler structure on the orbit space $\frac{\mu^{-1}(t)}{G}$.**

Proof: These two bilinear forms are transversal because they are G-invariant, and the symplectic form on M/G is closed by Marsden-Weinstein theorem. ■

COROLLARY: Let G be a compact group acting by Hamiltonian symplectomorphisms on an almost Kähler manifold. Assume that G preserves the almost complex structure. **Then the symplectic reduction $M/\!\!/G$ is also almost Kähler.** ■

REMARK: Since $K \subset T\mu^{-1}(t)$ is isotropic, one has $K \cap I(K) = 0$. This implies $T\mu^{-1}(t)/K = TM\Big|_{\mu^{-1}(t)}/K_{\mathbb{C}}$, where $K_{\mathbb{C}} = K \oplus I(K)$. **This is another way to equip the bundle $T\mu^{-1}(t)/K$ with an almost complex structure.**

CR-manifolds

DEFINITION: Let $B \subset TM$ be a sub-bundle equipped with a complex structure operator $I \in \operatorname{End} B$, $I^2 = -\operatorname{Id}$, and $B \otimes_{\mathbb R} \mathbb C = B^{1,0} \oplus B^{0,1}$ the corresponding eigenspace decomposition, $I\big|_{B^{1,0}} = \sqrt{-1}$, $I\big|_{B^{0,1}} = -\sqrt{-1}$. The pair (B, I) is called **a CR-structure on M** if $[B^{1,0}, B^{1,0}] \subset B^{1,0}$.

CLAIM: Let $Z \subset M$ be a submanifold of an almost complex manifold. Assume that $B := TZ \cap I(TZ)$ has constant rank. **Then $(B, I|_B)$ is a CR-structure on M.**

Proof: Any $(1,0)$-vector fields $X, Y \in B^{0,1} \subset TZ \otimes_{\mathbb R} \mathbb C$ can be smoothly extended to a section $\tilde X, \tilde Y \in T^{1,0}M$. A commutator $[\tilde X, \tilde Y]$ of $(1,0)$-vector fields on M is of type $(1,0)$. Commutator of vector fields tangent to Z remains tangent to Z. **Therefore, $[\tilde X, \tilde Y]\Big|_Z \in B^{1,0}$, giving $[X, Y] \in B^{1,0}$.** ■

CR-manifolds and leaf spaces

PROPOSITION 2: Let (Z,B,I) be a CR-manifold, equipped with an action of a compact group G compatible with the CR-structure. Assume that $TZ = B \oplus K$, where $K \subset TZ$ is the subspace generated by the vector fields tangent to the G-action. Then the operator I on B **defines a transversal complex structure** with respect to the foliation tangent to K. Morever, **the natural almost complex structure on Z/G induced by the action of I on $B = TZ/K = T(Z/G)$ is integrable.**

Proof: This operator is transversal because it is G-invariant, and integrable because $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. ■

Kähler reduction

COROLLARY: Let M be a Kähler manifold, G a compact Lie group freely acting on M by Hamiltonian isometries, $t \in \mathfrak{g}^*$ a central element, and $\mu : M \longrightarrow \mathfrak{g}^*$ an equivariant moment map. **Then the quotient $M /\!\!/ G := \frac{\mu^{-1}(t)}{G}$ is equipped with a natural Kähler structure.**

Proof. Step 1: Let $Z := \mu^{-1}(t)$ and let $K \subset TZ$ be the kernel of the restriction of the Kähler form to Z.

The space $Z := \mu^{-1}(t)$ is equipped with a CR-structure because Z is a subspace of a complex manifold with $TZ \cap I(TZ)$ of constant rank. Indeed, $I(TZ) \cap K = 0$ because K is isotropic, hence

$$\operatorname{rk}(TZ \cap I(TZ)) \leqslant \dim Z - \operatorname{rk} K = \dim M - 2 \operatorname{codim} Z.$$

However, the intersection $TZ \cap I(TZ)$ satisfies $\operatorname{rk}(TZ \cap I(TZ)) \geqslant \dim M - 2 \operatorname{codim} Z$ because it is an intersection of two subspaces of codimension $\operatorname{codim} Z$, giving $\operatorname{rk}(TZ \cap I(TZ)) = \dim Z - \operatorname{rk} K$.

Step 2: The quotient $M /\!\!/ G := Z/G$ is almost Kähler by Theorem 1. The almost complex structure on $M /\!\!/ G$ is integrable by Proposition 2. ■