Symplectic geometry

lecture 8: Ekeland-Hofer theorem (linear version deduced from non-squeezing)

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HSE, room 306, 16:20,

September 29, 2021

Gromov capacity (reminder)

DEFINITION: An (open) symplectic embedding is an open embedding of symplectic manifolds, symplectimorphic to its image.

DEFINITION: Let (M, ω) be a symplectic manifold, and r a supremum of radii of all symplectic balls of the same dimension, admitting a symplectic embedding to M. The number $\operatorname{capa}(M, \omega) := \pi r^2$ is called **Gromov symplectic capacity** of M.

THEOREM: (Ekeland-Hofer)
Let φ be an oriented diffeomorphism of symplectic manifold. Then **φ is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.**

Proof: Later today. ■

Ekeland-Hofer theorem (reminder)

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Let φ be an oriented diffeomorphism of symplectic manifold. Then φ **is a symplectomorphism if and only if φ preserves the symplectic capacity of all open subsets.**

Last lecture, the Ekeland-Hofer theorem **will be deduced from its linear version.**

THEOREM: Ekeland-Hofer, the linear version
Let $(V = \mathbb{R}^{2n}, \omega = \sum_i dp_i \wedge dq_i)$ be a symplectic vector space, and $\varphi : V \longrightarrow V$ an oriented linear map which preserves the Gromov capacity of all ellipsoids. **Then φ is a symplectomorphism.**

Proof: Later today. ■

Symplectic homeomorphisms

COROLLARY: Let $\varphi: B \longrightarrow \mathbb{R}^{2n}$ be an embedding from a symplectic ball to $(\mathbb{R}^n, \sum_i dp_i \wedge dq_i)$ which is locally a diffeomorphism. Assume that φ preserves the symplectic capacity of every ellipsoid E such that $\varphi(E)$ is convex. **Then φ is a symplectomorphism.** ■

The following theorem is deduced from Ekeland-Hofer.

THEOREM: (Eliashberg-Gromov)
Let (M, ω) be a symplectic manifold. Then **the group** Symp(M) **of symplectomorphisms is closed in the group** Diff(M) **of diffeomorphisms** with the open-compact topology.

Symplectic homeomorphisms

THEOREM: (Eliashberg-Gromov)
Let (M,ω) be a symplectic manifold. Then **the group** $\operatorname{Symp}(M)$ **of symplectomorphisms is closed in the group** $\operatorname{Diff}(M)$ **of diffeomorphisms** with the open-compact topology.

Proof: It would suffice to prove a weaker statement: let $\varphi_i: B \longrightarrow \mathbb{R}^{2n}$ be a sequence of symplectic embeddings, converging in C^0 to a smooth embedding $\varphi: B \longrightarrow \mathbb{R}^{2n}$. Then φ is a symplectomorphism.

Let $U \subset \varphi(B)$ be a convex set, such that $E := \varphi^{-1}(U)$ is convex, and $E_i := \varphi_i^{-1}(U)$. Then $\lim_i(d_h(E_i, E)) = 0$. Since φ_i converges to φ in C^0, we have $\lim_i d_H(\partial E, \partial E_i) = 0$. Therefore, for any $\varepsilon > 0$, almost all E_i satisfy $(1-\varepsilon)E \subset \partial E_i \subset (1+\varepsilon)E$. This implies that the symplectic capacity of E_i converges to $\operatorname{capa}_G(E)$. On the other hand, $\operatorname{capa}_G(E_i) = \operatorname{capa}_G(U)$, hence $\operatorname{capa}_G(U) = \lim_i \operatorname{capa}_G(E_i) = \operatorname{capa}_G(E)$, and φ_i preserves capacities. ■

DEFINITION: Symplectic homeomorphism is the closure of the group of symplectic diffeomorphisms in the group of homeomorphisms, with respect to the C^0-topology.

Symplectic ellipsoids

Claim 3: Let $V = \mathbb{R}^{2n}$, g a positive definite bilinear symmetric form on $\mathbb{R}^n$, and ω a non-degenerate antisymmetric 2-form. Then there exists a g-orthonormal basis v_i in V and a set of positive real numbers $\alpha_1, ..., \alpha_n$, independent from the choice of a basis, such that

$$\omega = \begin{pmatrix} \omega = \begin{matrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & \alpha_n \\ -\alpha_n & 0 \end{matrix} \end{pmatrix} \qquad (*)$$

In other words, $\omega = \sum_i \alpha_i x_i \wedge y_i$, where $x_1, y_1, x_2, y_2, ... \in V^*$ is the dual basis, and $\alpha_1 \geqslant \alpha_2 \geqslant ... \geqslant \alpha_n > 0$ positive real numbers.

Proof: Let $A := g^{-1}\omega$, that is, $g(Ax, y) = \omega(x, y)$. **The operator A is anti-symmetric: $g(Ax, y) = \omega(x, y) = -\omega(y, x) = -g(Ay, x) = -g(x, Ay)$.** An antisymmetric operator has a form (*) in some basis, which follows, for example, from the classification of rotations in the Euclidean space. ■

Symplectic ellipsoids (the second argument)

Another proof of Claim 3. Step 1:
Since $g(Ax,y)=\omega(x,y)=-\omega(y,x)=-g(Ay,x)=-g(x,Ay)$, we have $g(A^2x,y)=g(x,A^2y)$: the operator A^2 is symmetric. Since $g(A^2x,x)=-g(Ax,Ax)$, **it is symmetric and negative definite.**

Step 2: Let $g_A(x,y)=-g(A^2x,y)$, and $\{x_i\}$ be the basis in which both g and g_A are diagonal, with $g(x_i,x_i)=1$, $g_A(x_i,x_i)=b_i$. **Such a basis exists and is defined up to a map which is an isometry for both g and g_A.**

Step 3: Since $g(Ax_i,Ax_j)=b_i\delta_{ij}$ and $g_A(Ax_i,Ax_j)=-b_i^2\delta_{ij}$, the vectors $b_i^{-1/2}Ax_i$ are also orthogonal under g and g_A. For all i, take for x_{2i} the vector $b_{2i-1}^{-1/2}Ax_{2i-1}$. In this basis, A is written as

$$A=\begin{pmatrix} \omega = & 0 & \alpha_1 & & & \\ & -\alpha_1 & 0 & & 0 & \\ & & & \ddots & & \\ & & & & 0 & \alpha_n \\ & 0 & & & -\alpha_n & 0 \end{pmatrix},$$

where $\alpha_i=b_{2i-1}^{1/2}$. ■

Normal form of symplectic ellipsoid

DEFINITION: Let (V, ω) be a symplectic vector space. **Symplectic basis** is a basis $\{x_1, y_1, x_2, y_2, ...\}$ such that $\omega = \sum x_i^* \wedge y_i^*$ where $x_i^*, y_i^* \in V^*$ is the dual basis.

DEFINITION: **Normal form of an ellipsoid E in a symplectic vector space V** is $E = \{v = \sum a_i x_i + b_i y_i \ \mid \ \sum_i (a_i^2 + b_i^2)\alpha_i < 1\}$, where $\{x_i, y_i\}$ is a symplectic basis.

CLAIM: For any ellipsoid E in a symplectic vector space, **there exists a symplectic basis such that the ellipsoid is written in the normal form**, and the numbers α_i are determined uniquely by the ellipsoid and ω.

Proof: Let g be a bilinear symmetric form such that $E = \{v \in V \ \mid \ g(v, v) < 1\}$, and x_i', y_i' a g-orthonormal basis such that ω is written by the matrix (*). Then $x_i = x_i' \alpha_i^{-1/2}, y_i = y_i' \alpha_i^{-1/2}$ is a symplectic basis, and E is written as $\{v = \sum a_i x_i + b_i y_i \ \mid \ \sum_i (a_i^2 + b_i^2)\alpha_i < 1\}$. The numbers α_i are eigenvalues of $A = g\omega^{-1}$, hence they are determined uniquely. ∎

Symplectic ellipsoids and capacity

We compute the capacity of an ellipsoid using Gromov's non-squeezing theorem.

CLAIM: Let E be an ellipsoid in a symplectic space, written in the normal form $\{v = \sum a_i x_i + b_i y_i \ \ | \ \ \sum_i (a_i^2 + b_i^2)\alpha_i < 1\}$, and α_1 the smallest of the numbers α_i. **Then $\operatorname{capa}_G(E) = \pi\alpha_1^{-1}$.**

Proof: Consider the symplectic cylinder Cyl obtained as a product of the disk $\{ax_1 + by_1 \ \ | \ \ a^2 + b^2 \leqslant \alpha_1^{-1}\}$ and the vector space $\langle x_2, y_2, x_3, y_3, ...\rangle$. Then $\mathrm{Cyl} \supset E \supset B_r$, where B_r is a ball of radius $r := \alpha_1^{-1/2}$. **By Gromov's non-squeezing theorem, $\operatorname{capa}_G(\mathrm{Cyl}) = \operatorname{capa}_G(B_r) = \pi\alpha_1^{-1}$. By monotonicity of capa_G, its capacity is the same.** ■

Symplectic cylinders and capacity

DEFINITION: Let (V,ω) be a symplectic space, and $W \subset V$ a 2-dimensional symplectic space. Denote the symplectic orthogonal by $W^{\perp\omega}$. **Cylinder** $\operatorname{Cyl}_E \subset (V,\omega)$ is a product $\operatorname{Cyl}_E := E \times W^{\perp\omega}$ of an ellipsoid $E \subset W$ and $W^{\perp\omega}$. **We consider Cyl_E as a symplectic cylinder in V.**

The following claim follows from the result about the Gromov capacity of ellipsoids.

COROLLARY: Let $\varphi: V \longrightarrow W$ be an invertible linear map preserving the Gromov capacities of all ellipsoids. **Then φ preserves the symplectic capacities of cylinders.**

Proof: The cylinder $\operatorname{Cyl}_E := \{ax_1 + by_1 \ \mid\ a^2 + b^2 \leqslant r\}$ can be obtained as a union of an increasing family of ellipsoids

$$E_\alpha = \{v = \sum a_i x_i + b_i y_i \ \mid\ a_1^2 + b_1^2 + \sum_{i=2}^{\infty} \alpha(a_i^2 + b_i^2) < r\}, \alpha \longrightarrow \infty$$

of the same capacity. For any relatively compact symplectic ball $B_r \subset \operatorname{Cyl}_E$, it is contained in one of these ellipsoids, hence πr^2 is smaller than the capacity of the ellipsoid. Then $\operatorname{capa}_G(\operatorname{Cyl}_E) \leqslant \operatorname{capa}_G(E_\alpha) = \operatorname{capa}_G(E)$. ■

Fake cylinders

DEFINITION: Let $W_1 \subset V$ be a coisotropic subspace of codimension 2, and $W \subset V$ a complementary subspace, $W_1 \oplus W = V$. Consider an ellipsoid $E \subset W$. The product $W_1 \times E \subset V$ is called **a fake cylinder**.

THEOREM: A fake cylinder $Z := W_1 \times E \subset V$ is symplectomorphic to $\mathbb{R}^{2n}$ with the usual symplectic structure.

Proof. Step 1: Choose a symplectic basis in V in such a way that $W_1 = \langle x_1, x_2, x_3, y_3, x_4, y_4, ...\rangle$. Projecting E to $\langle y_1, y_2\rangle$ along W_1, we obtain an ellipsoid $E' \subset \langle y_1, y_2\rangle$. Clearly, $W_1 \times E = W_1 \times E'$, hence **we can assume that E is an ellipsoid in the space $\langle y_1, y_2\rangle$.**

Step 2: To finish the proof if would suffice to show that $Z = \langle x_1, x_2\rangle \times E$, where $E \subset \langle y_1, y_2\rangle$, is symplectomorphic to $\mathbb{R}^4$ with the usual symplectic structure.

Consider the projection $Z \longrightarrow E$ along $\langle x_1, x_2\rangle$. The form ω identifies the fibers of this projection with $T^*E = \langle y_1, y_2\rangle$. Therefore, **$Z$ is symplectomorphic to T^*E with the usual symplectic structure.** Since E is diffeomorphic to $\mathbb{R}^2$, the manifold Z is symplectomorphic to $T^*\mathbb{R}^2$. ■

Proof of the linear version of Ekeland-Hofer

COROLLARY: Let V, ω be a symplectic vector space, and $\varphi: V \longrightarrow V$ a linear map which preserves the symplectic capacity of all ellipsoids. **Then φ maps coisotropic spaces of codimension 2 to coisotropic spaces**.

Proof: Let $Z \subset V$ be a cylinder or a fake cylinder, $Z = W \times E$. The space W can be reconstructed from E as follows: it is a set of all $v \in V$ such that $\lambda v \in Z$ for all $\lambda \in \mathbb{R}$. It is coisotropic if and only if Z is a fake cylinder. **Since φ preserves the capacity, and capacity is finite for cylinders and infinite for fake cylinders, φ maps the cylinders to cylinders, and fake cylinders to fake cylinders**. ■

The linear version of Ekeland-Hofer is implied by the following lemma.

LEMMA: Let V, ω be a symplectic space, and $\varphi: V \longrightarrow V$ a linear map which takes coisotropic subspaces of codimension 2 to coisotropic subspaces. **Then $\varphi^* \omega = const \cdot \omega$, that is, φ is a composition of homothety and a symplectomorphsm.**

Linear maps preserving coisotropic spaces

LEMMA: Let V, ω be a symplectic space, and $\varphi: V \longrightarrow V$ a linear map which takes coisotropic subspaces of codimension 2 to coisotropic subspaces. **Then $\varphi^* \omega = const \cdot \omega$, that is, φ is a composition of homothety and a symplectomorphsm.**

Proof: Consider the correspondence mapping a coisotropic subspace $W \subset V$ of codimension 2 to $W^{\perp\omega}$. It is easy to see that W is coisotropic if and only if $\operatorname{Ann}(W)$ is isotropic. **We reduced the lemma to the following.**

LEMMA: Let V be a vector space, and ω_1, ω_2 symplectic forms such that any 2-dimensional plane which is isotropic in ω_1 is isotropic in ω_2. **Then ω_1, ω_2 are proportional**.

Proof: For all $x, y \in V$, we have $\omega_1(x,y) = 0 \Leftrightarrow \omega_2(x,y) = 0$. Multiplying ω_2 by a constant, we may assume that $\omega_1(a,b) = \omega_2(a,b) \neq 0$ for some $a, b \in V$. Then $\omega_1(a,c) = \omega_2(a,c)$ for any c; indeed, otherwise $\omega_1(a, c - tb) = 0$ for appropriate t, and $\omega_2(a,c) \neq 0$. Applying the same argument the second time, we obtain that $\omega_1(c,c') = \omega_2(c,c')$ for any $c, c' \in V$. ■