Complex geometry handout 6: Lie superalgebras

Definition 6.1. Let A^* be a graded commutative algebra, and $D: A^* \longrightarrow A^{*+i}$ a map which shifts the grading by i. It is called a **graded derivation**, or **superderivation**, if $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^j$.

Exercise 6.1. Prove that a supercommutator of superderivations is again a superderivation.

Exercise 6.2. Let $\tau: \Lambda^*(M) \longrightarrow \Lambda^{*-1}(M)$ be an odd derivation shifting the grading by -1. Prove that there exists a vector field $v \in TM$ such that $\tau = i_v$ (convolution with a vector field), or find a counterexample.

Exercise 6.3. Let $\tau: \Lambda^*(M) \longrightarrow \Lambda^{*-2}(M)$ be a derivation shifting the grading by -2. Prove that $\tau = 0$.

Definition 6.2. Let A^* be a graded commutative algebra over a field k. **Differential operators** on A^* are k-linear operators $D: A^* \longrightarrow A^*$ (even or odd), defined inductively as follows. **Differential operators of order 0** are maps $L_a(x) = ax$, where $a \in A^*$ (also even or odd). **Differential operators of order** p are maps $u: A^* \longrightarrow A^*$ such that $\{L_a, u\}$ is a differential operator of order $p-1$ for all $a \in A^*$.

Exercise 6.4. Let $D: A^* \longrightarrow A^*$ be a differential operator of order 1, and $a = D(1)$. Prove that $D - L_a$ is a super-derivation of A^*.

Exercise 6.5. Let $\omega \in \Lambda^2 V^*$ be a 2-form on a vector space V, $\nu \in \Lambda^2 V$ a bivector, $L_\omega(\eta) := \omega \wedge \eta$ and $\Lambda_\nu: \Lambda^i(V^*) \longrightarrow \Lambda^{i-2}(V^*)$ the convolution of a differential form and a bivector. Let $A \in \operatorname{End}(\Lambda^*(V^*))$ be the multiplication by a constant $-\Lambda_\nu(\omega)$. Prove that $[L_\omega, \Lambda_\nu] - A$ is an even derivation of $\Lambda^*(V^*)$.

Exercise 6.6. Let V be the fundamental representation of $\mathsf{Sp}(V)$, and $A \in \operatorname{End}(V)$ its automorphism commuting with $\mathsf{Sp}(V)$-action. Prove that A is a constant.

Exercise 6.7. Let $\omega \in \Lambda^2 V^*$ be a Hermitian 2-form on a n-dimensional complex vector space V, and L, Λ the corresponding Lefschetz operators (Lecture 11).

a. Prove that $[L,\Lambda]\big|_{\Lambda^1 V^*} = \alpha\, \mathsf{Id}$ for some scalar α.

b. Prove that $\Lambda(\omega) = n$. Deduce from this that $[L,\Lambda] + n$ is a derivation of $\Lambda^*(V^*)$.

c. Deduce that $n - [L,\Lambda]$ acts on k-forms as a multiplication by $k\alpha$, where α is a constant given by $[L,\Lambda]\big|_{\Lambda^1 V^*} + n = \alpha\, \mathsf{Id}$.

d. Prove that $[L,\Lambda]\big|_{\Lambda^{2n} V^*} = n\, \mathsf{Id}$.

e. Deduce that $[L,\Lambda]\Big|_{\Lambda^k V^*} = (k\alpha - n)\, \mathsf{Id}$, where $\alpha = 1$.

Remark 6.1. This gives another proof of the identity $[L,\Lambda]\Big|_{\Lambda^k V^*} = (n-k)\, \mathsf{Id}$.