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On Surfaces of Class VII_0 with Curves

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Introduction

The purpose of this article is to study surfaces of class VII_0. A compact complex surface is in class VII_0 if it is minimal and if its first Betti number is equal to one. Surfaces of class VII_0 with nonconstant meromorphic functions were completely classified by Kodaira [12] while surfaces of class VII_0 with second Betti number equal to zero were deeply studied by Inoue [4]. In this article we consider those surfaces of class VII_0 with positive second Betti number (hence having no meromorphic functions except constants). Now we have many examples of such surfaces – surfaces with global spherical shells [5, 6, 8, 19, 20]. Some of them are known as Inoue surfaces – parabolic, hyperbolic or half Inoue surfaces. Our main theorem is that *a surface of class VII_0 is a parabolic (or a hyperbolic, or a half) Inoue surface if it has an elliptic curve and a cycle of rational curves (or two cycles of rational curves, or a cycle of rational curves whose irreducible components generate the second homology group of the surface with rational coefficients)*. We also classify surfaces of class VII_0 with elliptic curves and surfaces of class VII_0 with b_2 equal to one having curves. The known characterizations of surfaces of class VII_0 are summarized in (10.3). This would be probably a modest step toward the complete classification of surfaces of class VII_0 with b_2 positive.

The most part of the results of this article have been announced in [19, 20]. The errors in [19] are corrected in (10.3).

The article is organized as follows. In §1 we recall the definitions of Inoue surfaces with b_2 positive. In §2, we study curves and their configurations on a surface of class VII_0. Any surface of class VII_0 has at most finitely many rational curves, each with at worst an ordinary double point, or finitely many nonsingular elliptic curves provided that it has no meromorphic functions except constants. The number of elliptic curves and cycles of rational curves on it is then at most two in total. In §3 we study meromorphic one forms with logarithmic poles with coefficients in flat line bundles. We verify two important lemmas (3.11) and (3.12). *In* §§4–6 *we consider mainly those surfaces of class* VII_0 *with either a pair of an elliptic curve and a cycle of rational curves or a pair of cycles of rational curves.* We show in §4 that the cycle(s) of rational curves is deformed into a nonsingular elliptic curve by deforming the surface. It is shown in §5 that a small deformation of the surface is a blown-up primary Hopf surface if it has two elliptic curves. Moreover we verify in §6 that there is a duality between two cycles of rationsl curves on the surface of class VII_0, if they exist – duality theorems (6.1), (6.8), (6.9). This may be viewed as a geometric explanation for part of the duality in [17]. §§7–9 are devoted to proving main theorems (7.1), (8.1) and (9.1) – characterizations of parabolic, hyperbolic or half Inoue surfaces. In §10 we classify surfaces of class VII_0 with elliptic curves. A classification table of surfaces of class VII_0 with curves is drafted in (10.3) by combining the results of Enoki [2] and this article. In §11 we classify surfaces of class VII_0 with b_2 equal to one having curves. In §12 some of the results in §4 and §5 are generalized. We show that any surface of class VII_0 with an anti-pluricanonical divisor is a (global) deformation of a blown-up primary Hopf surface.

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Notations

$\mathbf{C}$	(the ring of) complex numbers
$\mathbf{C}^*$	$\mathbf{C} - \{0\}$, (the group of) nonzero complex numbers
$\mathbf{R}$	(the ring of) real numbers
$\mathbf{Q}$	(the ring of) rational numbers
$\mathbf{Z}$	(the ring of) integers or the infinite cyclic group
$\mathbf{H}$	$\{z \in \mathbf{C};\ \mathrm{Im}(z) > 0\}$, the upper half plane
$S(t, n)$	a parabolic Inoue surface, see (1.1)
$S_\omega^{[n]}$	a hyperbolic Inoue surface, see (1.4)
$\hat{S}_\omega^{[2n+1]}$	a half Inoue surface, see (1.6)
S, S^*	compact complex surfaces

$\mathcal{O}_S$	the sheaf of germs over S of holomorphic functions
$\mathcal{O}_S^*$	the sheaf of germs over S of nonvanishing holomorphic functions
Ω_S^p	the sheaf of germs over S of holomorphic p-forms
Θ_S	the sheaf of germs over S of holomorphic vector fields
$\tilde{\Omega}_S$	see (3.11), (7.7)
$\Omega_S^1(\log D)$	the sheaf of germs over S of meromorphic 1-forms with logarithmic poles along D, see (3.1)
$\Theta_S(-\log D)$	$\mathscr{H}om_{\mathcal{O}_S}(\Omega_S^1(\log D), \mathcal{O}_S)$
$[D]$	the complex line bundle associated with a divisor D
$K, K_S; K_{\mathscr{S}}$	the canonical line bundle of S; of $\mathscr{S}$
$\mathcal{O}_S(-D)$	the sheaf of germs over S of holomorphic functions vanishing on (an effective divisor) D
$\mathcal{O}_D$	$\mathcal{O}_S/\mathcal{O}_S(-D)$
$\mathscr{F}_D, L_D$	$\mathscr{F} \otimes_{\mathcal{O}_S} \mathcal{O}_D$, $L \otimes_{\mathcal{O}_S} \mathcal{O}_D$ for a coherent sheaf $\mathscr{F}$ and a line bundle L
$h^q(S, \mathscr{F})$	$\dim_{\mathbf{C}} H^q(S, \mathscr{F})$ for a coherent sheaf $\mathscr{F}$ on S
$\chi(S, \mathscr{F})$	$\sum_{q=0}^{2} (-1)^q h^q(S, \mathscr{F})$
$h^{p,q}$	$h^q(S, \Omega_S^p)$
$b_i, b_i(S)$	the i-th Betti number of S
$b_2(C)$	$\#$ (irreducible components of C) for an effective divisor C
$\chi(S)$	$\sum_{i=0}^{4} (-1)^i b_i(S)$, the Euler number of S
$\pi_1(S)$	the fundamental group of S
$N(S), n(S)$	see (3.6)

§1. Inoue surfaces

The purpose of this section is to recall definitions of Inoue surfaces from [5, 6] for the use in §§7–9.

(1.1) *Parabolic Inoue surfaces.* Let $\mathscr{U}_k = \operatorname{Spec} \mathbf{C}[x_k, y_k]$, $\mathscr{V} = \operatorname{Spec} \mathbf{C}[w, w^{-1}, x]$ $(k \in \mathbf{Z})$. Define a complex manifold $\mathscr{X}$ by patching $\mathscr{U}_k$ and $\mathscr{V}$ by the relations;

$$x_{k+1} = y_k^{-1}, \quad y_{k+1} = x_k y_k^2,$$
$$w = x_k y_k, \quad x = x_k^{k+1} y_k^k,$$
$$x_k = w^{-k} x, \quad y_k = w^{k+1} x^{-1}.$$

Define a transformation $g(t)$ of $\mathscr{X}$ by

$$g(t)(\mathscr{U}_{k-1}) = \mathscr{U}_k, \quad g(t)(\mathscr{V}) = \mathscr{V},$$
$$g(t)\colon (x_{k-1}, y_{k-1}) \to (x_k', y_k') = (t^{-k} x_{k-1}, t^{k+1} y_{k-1}),$$
$$g(t)\colon (w, x) \to (w', x') = (tw, wx)$$

where $0 < |t| < 1$. Let $G_n = \{g(t)^{nm}; m \in \mathbf{Z}\}$. Then G_n acts properly discontinuously and freely so that we have a quotient $S(t^n, n) := \mathscr{X}/G_n$. We call $S(t^n, n)$ a parabolic Inoue surface. It turns out that $S(t^n, n)$ depends only on t^n and n (see (1.3)).

Let C_k be a nonsingular rational curve defined by

$$C_k \cap \mathscr{U}_{k-1}\colon x_{k-1} = 0, \quad C_k \cap \mathscr{U}_k\colon y_k = 0,$$

and let $C = \sum_{k \in \mathbf{Z}} C_k$. G_n transforms C onto C. The quotient $Z = C/G_n$ is a cycle of n rational curves. Let D be a divisor defined by

$$D \cap \mathscr{U}_k = \phi, \quad D \cap \mathscr{V}\colon x = 0.$$

Then D is transformed by G_n onto itself and the quotient $E := D/G_n$ is a nonsingular elliptic curve with $E^2 = -n$.

(1.2) **Theorem** [5]. *$S(t,n)$ is a* VII_0 *surface with $b_2 = n$ having no meromorphic functions except constants. There are an elliptic curve and a cycle of n rational curves on $S(t,n)$.*

(1.3) Let $\mathscr{X}$, $\mathscr{U}_k$, $\mathscr{V}$ and x_k, y_k, w, x be the the same as in (1.1). Then we define a transformation $g_n(t)$ of $\mathscr{X}$ as follows;

$$\begin{aligned} &g_n(t)\,(\mathscr{U}_{k-n}) = \mathscr{U}_k, \quad g_n(t)\,(\mathscr{V}) = \mathscr{V}, \\ &g_n(t)\colon (x_{k-n}, y_{k-n}) \to (x'_k, y'_k) = (t^{-nk} x_{k-n}, t^{n(k+1)} y_{k-n}), \\ &g_n(t)\colon (w, x) \to (w', x') = (t^n w, w^n x) \end{aligned}$$

where $0 < |t| < 1$. Let $G_n^* = \{g_n(t)^m;\ m \in \mathbf{Z}\}$. Then G_n^* acts on $\mathscr{X}$ properly discontinuously and freely so that we have a quotient $\mathscr{X}/G_n^*$ which turns out to be isomorphic to $S(t^n, n)$. In fact, defining a transformation h of $\mathscr{X}$ by

$$\begin{aligned} &h(\mathscr{U}_k) = \mathscr{U}_k, \quad h(\mathscr{V}) = \mathscr{V}, \\ &h\colon (x_k, y_k) \to (x'_k, y'_k) = (t^{-(n-1)k/2} x_k, t^{(n-1)(k+1)/2} y_k), \\ &h\colon (w, x) \to (w', x') = (t^{(n-1)/2} w, x), \end{aligned}$$

we have $g_n(t) \circ h = h \circ g(t)^n$. This h induces an isomorphism of $S(t^n, n)$ with $\mathscr{X}/G_n^*$.

(1.3) *Hyperbolic Inoue surfaces (Inoue-Hirzebruch surfaces).* Let ω be a totally real quadratic irrationality with $\omega > 1 > \omega' > 0$. We define $M(\omega) = \mathbf{Z} + \mathbf{Z}\omega$, $U(\omega) = \{x \in \mathbf{Q}(\omega);\ xM(\omega) = M(\omega)\}$, $U^+(\omega) = \{x \in U(\omega);\ x > 0, x' > 0\}$. It is known that $U^+(\omega)$ is a subgroup of $U(\omega)$ of index at most two, both $U(\omega)$ and $U^+(\omega)$ are infinite cyclic groups. We say that $M(\omega_1)$ and $M(\omega_2)$ are strictly equivalent if there exists $\delta \in \mathbf{Q}(\omega_1)$ such that $\delta M(\omega_1) = M(\omega_2)$, $\delta > 0$, $\delta' > 0$. By [6, Proposition (1.1)], there exist ω^* and $\beta_0 \in \mathbf{Q}(\omega)$ such that $\beta_0 > 0$, $\beta'_0 < 0$, $\beta_0 M(\omega) = M(\omega^*)$ and $\omega^* > 1 > \omega^{*\prime} > 0$. This $M(\omega^*)$ is unique up to strict equivalence (see ibid.). These β_0 and ω^* are given more explicitly as follows. Let the modified continued fraction expansions of ω and ω^{-1} be

$$\omega = [[\overline{n_0, n_1, \cdots, n_{r_0-1}}]]$$

$$\omega^{-1} = [[e_0, \cdots, e_{t-1}, \overline{m_0, \cdots, m_{s_0-1}}]]$$

where r_0 and s_0 are the smallest periods of the expansions. The integers n_λ and m_ν are greater than or equal to 2 and at least one n_λ and at least one m_ν are greater than

2. Let

$$
\begin{aligned}
\omega^* &= [[\overline{m_0, m_1, \cdots, m_{s_0-1}}]],\\
N_0 &= \begin{bmatrix} n_{r_0-1} & 1 \\ -1 & 0 \end{bmatrix} \cdots \begin{bmatrix} n_0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\\
N_0^* &= \begin{bmatrix} m_{s_0-1} & 1 \\ -1 & 0 \end{bmatrix} \cdots \begin{bmatrix} m_0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix},\\
B_0 &= \begin{bmatrix} e_{t-1} & 1 \\ -1 & 0 \end{bmatrix} \cdots \begin{bmatrix} e_0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.
\end{aligned}
$$

Then by [6, p. 93] we have

$$B_0 N_0 = N_0^* B_0, \quad \det B_0 = -1.$$

Let α_0 be an eigenvalue of N_0 with $\alpha_0 > 1$. Then

$$(\omega, 1) N_0 = \alpha_0 (\omega, 1), \quad (\omega^*, 1) N_0^* = \alpha_0 (\omega^*, 1).$$

Moreover α_0 is a generator of $U^+(\omega)$. By [6] there exists $\beta_0 \in \mathbf{Q}(\omega)$ such that

$$(\omega^*, 1) B_0 = \beta_0 (\omega, 1), \quad \beta_0 > 0 > \beta_0'.$$

Let $V_\lambda = \operatorname{Spec} \mathbf{C}[x_\lambda, y_\lambda]$, $W_\nu = \operatorname{Spec} \mathbf{C}[z_\nu, w_\nu]$ $(\lambda, \nu \in \mathbf{Z})$. We construct a complex manifold $\mathscr{U}$ by patching V_λ, $W_\nu (\lambda, \nu \in \mathbf{Z})$ by the relations;

$$
\begin{aligned}
&x_{\lambda+1} = y_\lambda^{-1}, \quad y_{\lambda+1} = x_\lambda y_\lambda^{n_\lambda},\\
&z_{\nu+1} = w_\nu^{-1}, \quad w_{\nu+1} = z_\nu w_\nu^{m_\nu},\\
&w_0 = y_0^e x_0^f, \quad z_0 = y_0^q x_0^h.
\end{aligned}
$$

Let C_λ and D_ν be nonsingular rational curves defined by

$$
\begin{aligned}
&C_\lambda \cap V_\lambda \colon x_\lambda = 0, \quad C_\lambda \cap V_{\lambda+1} \colon y_{\lambda+1} = 0,\\
&C_\lambda \cap V_\mu = C_\lambda \cap W_\mu = \phi \; (\mu \neq \lambda, \lambda+1; \mu, \lambda \in \mathbf{Z}),\\
&D_\nu \cap W_\nu \colon z_\nu = 0, \quad D_\nu \cap W_{\nu+1} \colon w_{\nu+1} = 0,\\
&D_\nu \cap W_\mu = D_\nu \cap V_\lambda = \phi \; (\mu \neq \nu, \nu+1; \lambda, \mu, \nu \in \mathbf{Z})
\end{aligned}
$$

and let $C = \sum_{\lambda \in \mathbf{Z}} C_\lambda$, $D = \sum_{\nu \in \mathbf{Z}} D_\nu$. Clearly C and D are infinite chains of nonsingular rational curves, $C_\lambda C_{\lambda+1} = D_\nu D_{\nu+1} = 1$, $C_\lambda^2 = -n_\lambda$, $D_\nu^2 = -m_\nu$, $C_\lambda C_\mu = D_\nu D_\mu = C_\lambda D_\nu = 0$ (otherwise). Let $p = |y_0^\omega x_0|$, $q = |y_0^{\omega'} x_0|$, $r = |w_0^{\omega^*} z_0|$, $s = |w_0^{\omega^{*\prime}} z_0|$. Then p and q are extended to continuous functions on $\mathscr{U} - D$ while r and s are extended to continuous functions on $\mathscr{U} - C$. Moreover we have $p^{\beta_0} = r$, $q^{\beta_0'} = s$. Therefore p and r can be extended to the whole $\mathscr{U}$. Denoting the extensions of p and r to $\mathscr{U}$ by the same letters, we let $\mathscr{D} = p^{-1}([0,1)) = r^{-1}([0,1))$. Then $\mathscr{D}$ is a simply connected open subset of $\mathscr{U}$. Define an automorphism g_0 of $\mathscr{U}$ by

$$
\begin{aligned}
&g_0(V_\lambda) \subset V_{\lambda - r_0}, \quad g_0(W_\nu) \subset W_{\nu - s_0},\\
&g_0 \colon (x_\lambda, y_\lambda) \to (x'_{\lambda - r_0}, y'_{\lambda - r_0}) = (x_\lambda, y_\lambda),\\
&g_0 \colon (z_\nu, w_\nu) \to (z'_{\nu - s_0}, w'_{\nu - s_0}) = (z_\nu, w_\nu).
\end{aligned}
$$

Then $g_0^*p = p^{\alpha_0}, g_0^*q = q^{\alpha_0'}, g_0^*r = r^{\alpha_0}, g_0^*s = s^{\alpha_0'}$ so that g_0 transforms $\mathcal{D}$ onto itself. Let $G_n = \{g_0^{nm}; m \in \mathbf{Z}\}$. The action of G_n on $\mathcal{D}$ is properly discontinuous and fixed-point free so that we have a quotient surface $\mathcal{D}/G_n$ which we denote by $S_\omega^{[n]}$ and call a *hyperbolic Inoue surface* or *an Inoue-Hirzebruch surface.* Let $\pi\colon \mathcal{D} \to S_\omega^{[n]}$ be the natural projection, $A_\lambda = \pi(C_\lambda)$, $B_\nu = \pi(D_\nu)$, $A = \pi(C)$ and $B = \pi(D)$. We have

$$A_0^2 = -n_0 + 2 \ (n = r_0 = 1), \quad A_\lambda^2 = -n_\lambda \ \text{(otherwise)},$$
$$B_0^2 = -m_0 + 2 \ (n = s_0 = 1), \quad B_\nu^2 = -m_\nu \ \text{(otherwise)}.$$

The curves A and B are cycles of nr_0 and ns_0 rational curves respectively. We define

$$\begin{aligned} \mathit{Zykel}\,(A) &= (n_0, \cdots, n_{r-1}), & \mathit{Zykel}\,(B) &= (m_0, \cdots, m_{s-1}) \\ &= (-A_0^2 + 2) \quad (n = r_0 = 1) & &= (-B_0^2 + 2) \quad (n = s_0 = 1) \\ &\quad (-A_0^2, \cdots, -A_{r-1}^2) & &\quad (-B_0^2, \cdots, -B_{s-1}^2) \\ &\quad \text{(otherwise)}, & &\quad \text{(otherwise)}. \end{aligned}$$

(1.5) **Theorem** [6]. *$S_\omega^{[n]}$ is a* VII$_0$ *surface with $b_2 = n(r_0 + s_0)$ having no meromorphic functions except constants. There are two cycles A and B of nr_0 and ns_0 rational curves on $S_\omega^{[n]}$.*

(1.6) *Half Inoue surfaces.* Consider the case where $[U(\omega)\colon U^+(\omega)] = 2$. Then we may take $\omega^* = \omega$, β_0 as β. Moreover $U(\omega)$ is generated by β_0 and we have $r_0 = s_0$, $N = N^*$, $B^2 = N$, $\beta_0^2 = \alpha_0$ [6, p. 93]. We define an automorphism τ_0 of $\mathcal{D}$ by

$$\tau_0(V_\lambda) \subset W_\lambda, \quad \tau_0(W_\nu) \subset V_{\nu - r_0},$$
$$\tau_0\colon (x_\lambda, y_\lambda) \to (z_\lambda', w_\lambda') = (x_\lambda, y_\lambda),$$
$$\tau_0\colon (z_\nu, w_\nu) \to (x_{\nu - r_0}', y_{\nu - r_0}') = (z_\nu, w_\nu).$$

Then the infinite cyclic group $\{\tau_0^k; k \in \mathbf{Z}\}$ operates on $\mathcal{D}$ freely and properly discontinuously. We denote by $\hat{S}_\omega^{[n]}$ the quotient of $\mathcal{D}$ by $\{\tau_0^{nk}; k \in \mathbf{Z}\}$ and *call it a half Inoue surface if n is odd.* It is easy to see that $\tau_0^2 = g_0$, $\hat{S}_\omega^{[2n]} \cong S_\omega^{[n]}$ (see (1.4)) and that τ_0^{2n+1} induces a fixed-point free involution i_{2n+1} of $\hat{S}_\omega^{[4n+2]}$ and $\hat{S}_\omega^{[4n+2]}/\{\text{id.}, i_{2n+1}\} \cong \hat{S}_\omega^{[2n+1]}$. The involution i_{2n+1} transforms a cycle of rational curves onto another on $\hat{S}_\omega^{[4n+2]}$. Therefore the half Inoue surface $\hat{S}_\omega^{[2n+1]}$ has a unique cycle C of rational curves with $C^2 < 0$ and $b_2(S) = \#$ (irreducible components of C) $= (2n+1)r_0$.

(1.7) **Theorem** [6]. *$\hat{S}_\omega^{[2n+1]}$ is a* VII$_0$ *surface with $b_2 = (2n+1)r_0$ having no meromorphic functions except constants. There is a unique cycle C of $(2n+1)r_0$ rational curves with $C^2 < 0$.*

§2. Curves on surfaces of class VII$_0$

(2.1) Let S be a surface of class VII$_0$ (or in short a VII$_0$ surface). It is by definition a compact complex surface with $b_1 = 1$ having no exceptional curves of the first kind. *We assume throughout this article, unless otherwise stated, that S has no meromorphic functions except constants.* Then we have, (2.1.1) $h^0(S, F) \leqq 1$ for any complex line bundle F on S.

We recall the numerical characters of S from [12, I, p. 755, II, p. 683].

(2.1.2) $h^{0,1} = 1$, $h^{1,0} = h^{2,0} = h^{0,2} = 0$, $-c_1^2 = c_2 = b_2$, $b_2^+ = 0$, $b_2^- = b_2$.

(2.2) **Lemma.** *Let D be an effective divisor on S. Then the following are true.*

(2.2.1) $h^1(D, \mathcal{O}_D) \leqq 2$. *If moreover D is reduced and connected, then* $h^1(D, \mathcal{O}_D) \leqq 1$.

(2.2.2) *If D is irreducible, then D is either a nonsingular rational curve, a rational curve with a node or a nonsingular elliptic curve.*

(2.2.3) $D^2 \leqq -2$ *for a nonsingular rational curve D.*

(2.2.4) *Let D_1 and D_2 be effective and reduced curves on S with no common components. Then $D_1 D_2 \leqq 2 - p_a(D_1) - p_a(D_2)$ and they meet transversally if $D_1 D_2 > 0$ where $p_a(D) = 1 + (KD + D^2)/2$.*

Proof. (2.2.1) From the exact sequence $0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$, we infer the exact sequence

$$\begin{aligned} 0 &\to H^0(S, \mathcal{O}_S) \to H^0(D, \mathcal{O}_D) \to H^1(S, \mathcal{O}_S(-D)) \\ &\to H^1(S, \mathcal{O}_S) \to H^1(D, \mathcal{O}_D) \to H^2(S, \mathcal{O}_S(-D)) \to 0 \end{aligned}$$

in view of (2.1.2). Therefore we have $h^1(D, \mathcal{O}_D) \leqq 1 + h^0(S, K+D) \leqq 2$ by (2.1.1). Assume next that D is reduced and connected and $h^1(D, \mathcal{O}_D) = 2$. Since $b_1 = 1$, we have an n-fold unramified covering $\pi: S^* \to S$ of S for arbitrary n. Then S^* is a VII_0 surface with no meromorphic functions except constants. Let $n = 2$. Let K^* be the canonical line bundle of S^* and $D^* = \pi^*(D)$. Hence $(K^* + D^*)D^* = 2h^1(D^*, \mathcal{O}_{D^*}) - 2h^0(D^*, \mathcal{O}_{D^*}) \leqq 2$ in view of the first assertion of (2.2.1). However we have $(K^* + D^*)D^* = 2(K+D)D = 4$, which is absurd.

(2.2.2) Assume that D is a rational curve with a cusp. Take a nontrivial triple covering $\pi: S^* \to S$ of S and let $D^* = \pi^*(D)$. Since D is simply connected, we have $h^1(D^*, \mathcal{O}_{D^*}) = 3$ which contradicts (2.2.1). In view of (2.2.1), $h^1(D, \mathcal{O}_D) = 0$ or 1 so that (2.2.2) follows.

(2.2.3) If D is an effective divisor, then $D^2 \leqq 0$ by (2.1.2). Assume that D is a nonsingular rational curve with $D^2 = 0$. Since $\mathcal{O}_D(D) = \mathcal{O}_D$, we infer $h^1(S, \mathcal{O}_S(D)) = 0$ from the exact sequence

$$\begin{aligned} 0 &\to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(D)) \to H^0(D, \mathcal{O}_D(D)) \\ &\to H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S(D)) \to H^1(D, \mathcal{O}_D(D)) \to 0. \end{aligned}$$

Therefore we have the exact sequence

$$0 \to H^0(S, \mathcal{O}_S(D)) \to H^0(S, \mathcal{O}_S(2D)) \to H^0(D, \mathcal{O}_D(2D)) \to H^1(S, \mathcal{O}_S(D)) = 0$$

where $h^0(S, \mathcal{O}_S(D)) = h^0(S, \mathcal{O}_S(2D)) = h^0(D, \mathcal{O}_D(2D)) = 1$. This is a contradiction. If $D^2 = -1$, then D is an exceptional curve of the first kind. Such D does not exist by our assumption.

(2.2.4) If $D_1 D_2 = 0$, then it is clear from (2.2.1). If $D_1 D_2 > 0$, then $D_1 + D_2$ is reduced and connected so that $p_a(D) = h^1(D, \mathcal{O}_D) \leqq 1$ where $D = D_1 + D_2$.

Therefore

$$D_1D_2 = p_a(D_1+D_2) - p_a(D_1) - p_a(D_2) + 1 \leqq 2 - p_a(D_1) - p_a(D_2).$$

Now we suppose that D_1 and D_2 meet at a point with multiplicity $\geqq 2$. Then we have $D_1D_2 = 2$, $p_a(D_1) = p_a(D_2) = 0$, $p_a(D) = h^1(\mathcal{O}_D) = 1$ and that $D_1 + D_2$ is simply connected. Let $\pi\colon S^* \to S$ be a triple covering of S, $D^* = \pi^*D$. Then D^* is a disjoint union of 3 copies of $D_1 + D_2$. Hence we have $h^1(D^*, \mathcal{O}_{D^*}) = 3h^1(D, \mathcal{O}_D) = 3$, which is a contradiction. Q.E.D.

(2.3) **Lemma.** *Let D be a reduced connected effective divisor with $h^1(D, \mathcal{O}_D) = 1$. Assume that $h^1(D', \mathcal{O}_{D'}) = 0$ for any proper subcurve D' of D. Then D is either a nonsingular elliptic curve or a rational curve with a node or a cycle of nonsingular rational curves.*

Proof. If D is irreducible, then (2.3) follows from (2.2.2). Let $D = \sum_{\nu=1}^{n} D_\nu$ be the decomposition into irreducible components and assume $n \geqq 2$. Then by our assumption D_ν is a nonsingular rational curve for any ν. If $D_\nu(D - C_\nu) = 1$ or 0 for some ν, then we have $h^1(D', \mathcal{O}_{D'}) = 1$ where $D' = D - D_\nu$. Hence $D_\nu(D - D_\nu) \geqq 2$. Since $p_a(D) = 1$, we have

$$1 = \sum_{\nu=1}^{n} p_a(D_\nu) + \sum_{\nu<\mu} D_\nu D_\mu + (1-n), \quad \text{i.e.} \quad \sum_{\nu<\mu} D_\nu D_\mu = n.$$

Therefore

$$2n = 2\sum_{\nu<\mu} D_\nu D_\mu = \sum_{\nu=1}^{n} D_\nu(D-D_\nu) \geqq 2\mathrm{n}$$

hence $D_\nu(D-D_\nu) = 2$. If $n = 2$, then $D = D_1 + D_2$, D_1 and D_2 meet at two points transversally by (2.2.4). If $n \geqq 3$, then by reordering D_ν suitably, we have $D = \sum_{\nu=1}^{n} D_\nu$, $D_\nu D_{\nu+1} = D_1 D_n = 1$, $D_\nu D_\mu = 0$ (otherwise and $\nu \neq \mu$). Q.E.D.

(2.4) A divisor D is said to be *a cycle of rational curves* if D is either a rational curve with a node or a cycle of nonsingular rational curves.

We notice that *a surface of class* VII_0 *with a cycle of rational curves has no meromorphic functions except constants.*

(2.5) **Lemma.** *A line bundle F $(\in H^1(S, \mathcal{O}_S^*))$ is flat, that is, $F \in H^1(S, \mathbf{C}^*)$ if and only if $F^2 = 0$.*

Proof. Only if part is clear. We prove if part. By the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[d, equal] & \mathbf{C} \arrow[r, "\exp(2\pi i\cdot)"'] \arrow[d] & \mathbf{C}^* \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \mathbf{Z} \arrow[r] & \mathcal{O}_S \arrow[r, "\exp(2\pi i\cdot)"'] & \mathcal{O}_S^* \arrow[r] & 1
\end{tikzcd}$$

we have exact sequences

$$\begin{tikzcd}
H^1(S,\mathbf{C}) \arrow[r] \arrow[d, "j"] & H^1(S,\mathbf{C}^*) \arrow[r, "c"] \arrow[d, "i"] & H^2(S,\mathbf{Z}) \arrow[r, "h"] \arrow[d, "k"] & H^2(S,\mathbf{C}) \\
H^1(S,\mathcal{O}_S) \arrow[r] & H^1(S,\mathcal{O}_S^*) \arrow[r, "c"] & H^2(S,\mathbf{Z}) &
\end{tikzcd}$$

We assume that $F^2=0$. In view of (2.1.2) we have $c(F)_{\mathbf{R}}=0$ so that $c(F)_{\mathbf{C}}=0$, that is, $hk^{-1}c(F)=0$. Therefore there exists an element G in $H^1(S,\mathbf{C}^*)$ such that $k^{-1}c(F)=c(G)$, i.e. $c(F)=c(i(G))$. Since j is an isomorphism, this implies that $F-i(G)=\exp(2\pi i(j(H)))$ for an element H of $H^1(S,\mathbf{C})$. Thus $F=i(G)+\exp(2\pi i(j(H)))$ is in the image of $H^1(S,\mathbf{C}^*)$. Q.E.D.

(2.6) **Lemma.** *Let D be an effective divisor with $h^1(D,\mathcal{O}_D)\geqq 1$. Then the restriction map $r\colon H^1(S,\mathcal{O}_S)\to H^1(D,\mathcal{O}_D)$ is injective.*

Proof. First we shall prove that $h^1(D_{\mathrm{red}},\mathcal{O}_{D_{\mathrm{red}}})\geqq 1$ where D_{red} is the reduced divisor with the same support as D. If $h^1(D_{\mathrm{red}},\mathcal{O}_{D_{\mathrm{red}}})=0$, then any connected component D' of D_{red} satisfies $h^1(D',\mathcal{O}_{D'})=0$. This implies that D' is simply connected. Letting $\pi\colon S^*\to S$ be a triple covering of S, $D^*=\pi^*(D)$, we have $h^1(D^*,\mathcal{O}_{D^*})\geqq 3$ which is a contradiction of (2.2.1). Hence we may assume D to be connected and reduced. Then D is either a nonsingular elliptic curve or a cycle of rational curves, in view of (2.3). Let i be the inclusion mapping of D into S, $i_*\colon H_1(D,\mathbf{C})\to H_1(S,\mathbf{C})$ the induced homomorphism. Then i_* is surjective. In fact, otherwise, we can choose an element of $H_1(S,\mathbf{Z})$ of infinite order not contained in $i_*(H_1(D,\mathbf{Z}))$. From this we can construct a triple covering $\pi\colon S^*\to S$ of S such that $\pi^{-1}(D)$ is the disjoint union of three copies of D. This contradicts (2.2.1). Therefore i_* is surjective so that $i^*\colon H^1(S,\mathbf{C})\to H^1(D,\mathbf{C})$ is injective. If D is a cycle of rational curves, then $H^1(D,\mathcal{O}_D)\cong H^1(D,\mathbf{C})$. Since $H^1(S,\mathbf{C})\cong H^1(S,\mathcal{O}_S)$, the homomorphism r is injective. Next we consider the case where D is a nonsingular elliptic curve. Since i^* is injective and $H^1(S,\mathcal{O}_S)$ is mapped into $H^1(D,\mathcal{O}_D)\cong H^{0,1}\subset H^1(D,\mathbf{C})$ by r, we have $r(H^1(S,\mathcal{O}_S))=H^1(D,\mathcal{O}_D)$. So r is injective. Q.E.D.

(2.7) **Lemma.** *Let D be an effective divisor on S. Then the following are equivalent.*

(2.7.1) $H^1(D_{\mathrm{red}},\mathcal{O}_{D_{\mathrm{red}}})=0$,

(2.7.2) $H^1(D,\mathcal{O}_D)=0$,

(2.7.3) $H^1(F,\mathcal{O}_F)=0$ *for any effective divisor with* $\operatorname{supp}(F)\subset\operatorname{supp}(D)$,

(2.7.4) $p_a(F)\leqq 0$ *for any effective divisor F with* $\operatorname{supp}(F)\subset\operatorname{supp}(D)$.

Proof. By the proof of (2.6), (2.7.1) and (2.7.2) are equivalent. Let F be an effective divisor with $\operatorname{supp}(F)\subset\operatorname{supp}(D)$. Assume that $h^1(F,\mathcal{O}_F)\geqq 1$. Then it follows that $h^1(F_{\mathrm{red}},\mathcal{O}_{F_{\mathrm{red}}})\geqq 1$ hence $h^1(D_{\mathrm{red}},\mathcal{O}_{D_{\mathrm{red}}})\geqq 1$ and $h^1(D,\mathcal{O}_D)\geqq 1$. Thus (2.7.3) follows from both (2.7.1) and (2.7.2). In view of [1, Theorem 1.7], (2.7.3) and (2.7.4) are equivalent. The remaining implication is evident. Q.E.D.

(2.8) **Lemma.** *Assume that D is an effective divisor on S such that $h^1(D, \mathcal{O}_D) = 2$ and $h^1(D', \mathcal{O}_{D'}) \leqq 1$ for any proper subcurve D' of D. Then we have $K_S + D = 0$.*

Proof. By (2.2.1), we have $h^0(S, K_S + D) = 1$. Hence there exists an effective divisor E linearly equivalent to $K_S + D$. If D and E have an irreducible component F in common, let $D' = D - F$. Then $h^1(D', \mathcal{O}_{D'}) \geqq 1$ since $h^2(S, \mathcal{O}_S(-D')) = 1$. In view of (2.6), we have $h^1(D', \mathcal{O}_{D'}) = 2$ which contradicts the assumption. Hence D and E have no irreducible components in common. By assuming that $E \neq 0$, we shall derive a contradiction. Assume that $E^2 = 0$. Then E is flat in view of (2.5) so that $p_a(E) = 1$ and $DE = 0$. By (2.7) we have $h^1(E, \mathcal{O}_E) \geqq 1$, hence $h^1(E+D, \mathcal{O}_{E+D}) \geqq 3$ which is a contradiction. Hence $E^2 < 0$. Therefore $KE = E^2 - DE < 0$ so that $KE_\nu < 0$ for at least one irreducible component E_ν of E. Since $E_\nu^2 \leqq 0$, we have $p_a(E_\nu) = 0$. In view of (2.2.3), $p_a(E_\nu) = 1 + (KE_\nu + E_\nu^2)/2 < 0$ which is a contradiction. Q.E.D.

(2.9) **Lemma.** *Let D be an effective divisor with $h^1(D, \mathcal{O}_D) = 2$, $h^1(D', \mathcal{O}_{D'}) \leqq 1$ for any proper subcurve D' of D, E an effective connected reduced divisor having no irreducible components in common with D. Then $ED = 0$ and E is a tree of nonsingular rational curves with intersection graph given by A_n, D_n, E_6, E_7 and E_8.*

Proof. Let F be an effective divisor ($\neq 0$) with $\operatorname{supp}(F) \subset \operatorname{supp}(E)$. If $F^2 = 0$, then F is flat in view of (2.5) so that $p_a(F) = 1$, $DF = 0$, and $h^1(F, \mathcal{O}_F) \geqq 1$ by (2.7). Therefore $h^1(F+D, \mathcal{O}_{F+D}) \geqq 3$ which is a contradiction. Hence $F^2 < 0$. This implies that the intersection matrix of E is negative definite. Let F be an irreducible component of E. We have $F^2 < 0$ and $KF = -DF \leqq 0$ by (2.8). Hence $p_a(F) = 0$, $KF = -DF = 0$, $F^2 = -2$ in view of (2.2.3). Therefore $DE = 0$. The final assertion follows from [1]. Q.E.D.

(2.10) **Lemma.** *Let D be a divisor ($\neq 0$) such that the line bundle $[D]$ associated to D is flat. Then $D = mE$ or $mE + nF$ $(m, n \neq 0)$ where E and F are one of the following,*

(2.10.1) *a nonsingular elliptic curve G with $G^2 = 0$,*

(2.10.2) *a rational curve G with a node and with $G^2 = 0$,*

(2.10.3) *a cycle $G = G_1 + G_2$ of rational curves with $G_1G_2 = 2$, $G_\nu^2 = -2$.*

(2.10.4) *a cycle $G = \sum_{\nu=1}^{r} G_\nu$ of nonsingular rational curves G_ν with $G_\nu G_{\nu+1} = G_1 G_r = 1$, $G_\nu^2 = -2$, $G_\nu G_\mu = 0$ (otherwise) $(r \geqq 3)$.*

Proof. First we assume that D is effective and connected. If D_{red} is irreducible, then $(D_{\mathrm{red}})^2 = 0$, $p_a(D_{\mathrm{red}}) = 1$ so that (2.10) follows from (2.2). If D_{red} is reducible, let $D = \sum_{\nu=1}^{r} n_\nu D_\nu$ be the decomposition of D into irreducible components. Since $KD_\nu = 2p_a(D_\nu) - D_\nu^2 - 2$, $D_\nu^2 \leqq 0$, we have $KD_\nu \geqq 0$ in view of (2.2). On the other hand

$$1 = p_a(D) = KD/2 + 1 = \left(\sum_{\nu=1}^{r} n_\nu KD_\nu\right)/2 + 1 \geqq 1 .$$

This implies that $KD_\nu = 0$. Since $-n_\nu D_\nu^2 = D_\nu\left(\sum_{\mu \neq \nu} n_\mu D_\mu\right) > 0$, we have $D_\nu^2 < 0$. Hence $D_\nu^2 = -2$, $p_a(D_\nu) = 0$. Thus we have

$$2n_\nu = \sum_{\mu \neq \nu} n_\mu D_\nu D_\mu .$$

This implies that D is one of the curves appearing as singular fibres in a pencil of elliptic curves ([11, p. 567]). From [11, p. 565] together with a remark that any effective divisor E on S having a simply connected support satisfies $h^1(E, \mathcal{O}_E) = 0$, we infer that D is one of ${}_mI_b$. Next we consider the general case. First we assume that D is effective. Let $D = \sum_{i=1}^{s} D^{(i)}$ be the decomposition of D into connected components. Then it is easy to see that $[D^{(i)}]$ is flat. Therefore $D^{(i)}$ is one of the above curves (2.10.1) ~ (2.10.4) as we have shown. Finally we consider the case where neither D nor $-D$ are effective. We write $D = E - F$ for effective divisors E and F with no common components. Since $D^2 = 0$, we have $0 = E^2 + F^2 - 2EF$. On the other hand, we have $E^2 \leqq 0$, $F^2 \leqq 0$ and $-EF \leqq 0$. This implies that $E^2 = F^2 = EF = 0$. From what we have seen, any connected component of E and F is one of the curves (2.10.1) ~ (2.10.4). In view of (2.2.1) the number of connected components is in any case not greater than 2.

(2.11) **Lemma.** *Let D_1 and D_2 be connected reduced divisors on S with $h^1(D_\lambda, \mathcal{O}_{D_\lambda}) = 1$, $\lambda = 1, 2$, $D_1 \cap D_2 = \emptyset$. Then D_1 is a nonsingular elliptic curve iff $D_2^2 = 0$.*

The following is an application of Ma. Kato's proof of (2.12.1).

Proof. By (2.3) D_λ is either a nonsingular elliptic curve or a cycle of rational curves. Suppose that D_1 is a nonsingular elliptic curve. Then it follows that $\mathrm{Pic}^0(D_1) \cong D_1$. Since $H^1(S, \mathbf{C}^*) \cong \mathrm{Hom}(H_1(S, \mathbf{Z}), \mathbf{C}^*) \cong \mathbf{C}^* \oplus$ (finite group), there exists a line bundle $F (\in H^1(S, \mathbf{C}^*))$ of infinite order such that F_{D_1} (:= the restriction of F to D_1) is trivial. (This is a remark by Inoue.) Let $D = D_1 + D_2$. We shall show $H^0(S, F) \neq 0$ or $H^0(S, -F) \neq 0$. Assume $H^0(S, -F) = 0$. Then we consider the exact sequence

$$\begin{aligned} 0 &\to H^0(S, -F - [D]) \to H^0(S, -F) \to H^0(D, -F_D) \\ &\to H^1(S, -F - [D]) \to H^1(S, -F) \to H^1(D, -F_D) \\ &\to H^2(S, -F - [D]) \to H^2(S, -F) \to 0 . \end{aligned}$$

We have $H^0(S, -F - [D]) = 0$ by assumption. In view of (2.8), we have $K_S + D = 0$ so that $\chi(S, -F - [D]) = \chi(S, F) = 0$, and

$$h^2(S, -F - [D]) = h^1(S, -F - [D]) \geqq h^0(D, -F_D) \geqq h^0(D_1, \mathcal{O}_{D_1}) = 1 .$$

Hence we have $h^2(S, -F - [D]) = h^0(S, F) = 1$, $h^0(D_2, -F_{D_2}) = 0$. Let E be the effective divisor $[E] = F$. In view of (2.10) we write $E = m_1 E_1 + m_2 E_2$ ($m_1 > 0$, $m_2 \geqq 0$) with $E_\nu^2 = 0$. If $m_2 > 0$, then by (2.2) we may assume that $D_1 = E_1$, $D_2 = E_2$, hence $D_2^2 = 0$. If $m_1 > 0$, $m_2 = 0$, then either $E_1 = D_1$ or $E_1 = D_2$. We shall show $E_1 = D_2$. Indeed, otherwise, $D_1^2 = E_1^2 = 0$ and $[D_1]_{D_2}$ is trivial by (2.5). Hence $F = [m_1 D_1]$ is contained in $\mathrm{Ker}(H^1(S, \mathbf{C}^*) \to H^1(D_2, \mathcal{O}_{D_2}^*))$ which contradicts $h^0(D_2, -F_{D_2}) = 0$. Therefore we have $D_2 = E_1$, $D_2^2 = E_1^2 = 0$. Next we shall show if part. Assume $D_2^2 = 0$. Then $[D_2]$ is flat in view of (2.5). Since $D_1 D_2 = 0$, $[D_2]_{D_1}$ is trivial. This shows that the restriction homomorphism $r\colon H^1(S, \mathbf{C}^*) \to H^1(D_1, \mathcal{O}_{D_1}^*)$ has the non-trivial kernel of infinite order. If D_1 is a cycle of rational curves, then by the proof of (2.6) Ker (r) is finite. Therefore D_1 is a nonsingular elliptic curve. Q.E.D.

(2.12) **Lemma.** *Let D be a reduced effective divisor such that $h^1(D, \mathcal{O}_D) = 2$ and $h^1(D', \mathcal{O}_{D'}) \leqq 1$ for any proper subcurve D' of D. Then $D = D_1 + D_2$, $D_1 D_2 = 0$, D_ν being a connected component of D, $D_1 \neq D_2$. And one of the following holds;*

(2.12.1) *both D_1 and D_2 are nonsingular elliptic curves with $D_1^2 = D_2^2 = 0$,*

(2.12.2) *D_1 is a nonsingular elliptic curve with $D_1^2 < 0$, and D_2 is a cycle of rational curves with $D_2^2 = 0$,*

(2.12.3) *both D_1 and D_2 are cycles of rational curves with $D_1^2 < 0$, $D_2^2 < 0$.*

Clear from (2.2), (2.9) and (2.11). (2.12.1) was proved by Kato.

(2.13) **Lemma.** *Let C be a cycle of rational curves on S. Then we have $[H_1(S, \mathbf{Z})\colon i_* H_1(C, \mathbf{Z})] = 1$ or 2 where i is the inclusion map of C into S.*

Proof. Let N be the cokernel of i_*, the following is exact, $0 \to H_1(C, \mathbf{Z}) \xrightarrow{i_*} H_1(S, \mathbf{Z}) \to N \to 0$. Then in view of the universal coefficient theorem and $H_1(C, \mathbf{Z}) \cong \mathbf{Z}$, the following is exact,

(2.13.1)
$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(N, \mathbf{C}^*) \arrow[r] & \operatorname{Hom}(H_1(S, \mathbf{Z}), \mathbf{C}^*) \arrow[r] \arrow[d, phantom, "\wr\|"] & \operatorname{Hom}(H_1(C, \mathbf{Z}), \mathbf{C}^*) \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\ & & H^1(S, \mathbf{C}^*) \arrow[r, "i^*"] & H^1(C, \mathbf{C}^*) & \end{tikzcd}$$

Since N is finite and abelian, we have $\operatorname{Hom}(N, \mathbf{C}^*) \cong N$. If N has an element of order n, then we have a flat line bundle F on S of order n such that F_C is trivial. From this F we can construct an n-sheeted unramified covering $\pi\colon S' \to S$ such that $\pi^{-1}(C)$ is a disjoint union of n-copies of C. Since S' is a surface of class VII_0 with no meromorphic functions, we have $n \leqq 2$ by (2.2.1). If N is of order n, we can construct in the similar manner an unramified abelian covering $\pi\colon S' \to S$ such that the Galois group of π is N and $\pi^{-1}(C)$ is a disjoint union of n copies of C. Therefore $n \leqq 2$. Q.E.D.

(2.14) **Corollary.** *Let C be a cycle of rational curves on S. Suppose that*

(2.14.1) *there exists an irreducible curve E with $EC > 0$, or that*

(2.14.2) *there exists an elliptic curve or another cycle of rational curves.*

Then $H_1(S, \mathbf{Z}) = i_* H_1(C, \mathbf{Z})$.

Proof. Suppose that the order of N is 2 with the notations in (2.13). Then we have a double covering $\pi\colon S' \to S$ with $\pi^{-1}(C)$ two copies of C. Hence by (2.2), (2.8) and (2.9) we have a contradiction of (2.14.1) or (2.14.2). Q.E.D.

§3. Meromorphic one-forms with logarithmic poles

(3.1) Let D be an effective divisor on S. Then we define a locally free sheaf $\Omega_S^1(\log D)$ by

$$\Omega_S^1(\log D) = \{\omega \in \Omega_S^1(D_{\mathrm{red}});\ d\omega \in \Omega_S^2(D_{\mathrm{red}})\}$$

where $\Omega_S^q(D_{\mathrm{red}})$ stands for the sheaf of germs of meromorphic q-forms with poles of at most order one at D. Let p be a nonsingular point of D, x and y local parameters with p center such that D is defined by $x = 0$ at p. Then $\Omega_S^1(\log D)$ is an $\mathcal{O}_S$ module generated by dy and $x^{-1}dx$ at p. If p is a singular point of D, we can take in view of (2.2) local parameters x and y with p center such that D: $xy = 0$. Then $\Omega_S^1(\log D)$ is an $\mathcal{O}_S$ module generated by $x^{-1}dx$ and $y^{-1}dy$ at p.

(3.2) Since there are only finitely many irreducible curves on a surface with no meromorphic functions except constants, we let M be the maximal reduced divisor on S, $M = \sum_{j=1}^{p} M_j$ the decomposition of M into irreducible components. Denoting by $\tilde{M}$ the normalisation of M, we have the following exact sequence;

$$0 \to \Omega_S^1 \to \Omega_S^1(\log M) \to \mathcal{O}_{\tilde{M}} \to 0\,.$$

(3.3) **Lemma.** *The following is commutative and the first row is exact*:

$$\begin{tikzcd}
0 \arrow[r] & H^0(S, \Omega_S^1(\log M)) \arrow[r] & H^0(\tilde{M}, \mathbf{C}) \arrow[r, "\delta"] & H^2(S, \mathbf{C}) \\
 & & H^0(\tilde{M}, \mathbf{Z}) \arrow[u] \arrow[r, "\delta_{\mathbf{Z}}"] & H^2(S, \mathbf{Z}) \arrow[u]
\end{tikzcd}$$

where $\delta([M_j]^*) = c(M_j)$, $[M_j]^*$ *standing for a function whose support is* M_j *and which takes the value* 1.

Proof. From (3.2) we infer an exact sequence

$$0 \to H^0(S, \Omega_S^1(\log M)) \to H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}) \to H^1(S, \Omega_S^1) \to \cdots\,.$$

We shall show $H^1(S, \Omega_S^1) \cong H^2(S, \mathbf{C})$. Consider the exact sequences

$$0 \to \mathbf{C} \to \mathcal{O}_S \to d\mathcal{O}_S \to 0$$
$$0 \to d\mathcal{O}_S \to \Omega_S^1 \to \Omega_S^2 \to 0\,.$$

Since $h^{0,2} = 0$ and $H^1(S, \mathbf{C}) \cong H^1(S, \mathcal{O}_S)$, we infer $H^q(S, d\mathcal{O}_S) \cong H^{q+1}(S, \mathbf{C})$ $(q \geqq 1)$ and the following exact sequence

$$0 \to H^1(S, d\mathcal{O}_S) \to H^1(S, \Omega_S^1) \to H^1(S, \Omega_S^2) \to H^2(S, d\mathcal{O}_S) \to 0\,.$$

By (2.1.2) we have $h^{2,1} = h^{0,1} = 1$, $h^2(S, d\mathcal{O}_S) = b_3 = 1$. Therefore $H^1(S, \Omega_S^1) \cong H^2(S, \mathbf{C})$. It is easy to see that

$$\delta([M_j]^*) = c(M_j) \text{ (the first Chern class of } M_j)$$

so that δ comes from the $\mathbf{Z}$ homomorphism $\delta_{\mathbf{Z}}$: $H^0(\tilde{M}, \mathbf{Z}) \to H^2(S, \mathbf{Z})$. Q.E.D.

(3.4) **Lemma.** *Let M be the maximal reduced divisor on S. Then $H^0(S, \Omega_S^1(\log M))$ is generated by divisors* (2.10.1) $\sim$ (2.10.4).

Proof. In view of (3.3), $H^0(S, \Omega_S^1(\log M))$ is generated by $\operatorname{Ker}(\delta_Z)$. By (2.5) and (3.3) $\operatorname{Ker}(\delta_Z)$ is generated by divisors whose associated line bundles are flat. Our lemma follows from (2.10). Q.E.D.

(3.5) **Theorem** (Ma. Kato). *The number $\rho_r(S)$ of irreducible rational curves on S is not greater than b_2.*

Proof. In view of (2.10), (3.3) and (3.4) $\rho_r(S) \leqq 2 + b_2(S)$. Let S^* be a triple unramified covering of S. It is easy to see that $\rho_r(S^*) = 3\rho_r(S)$, $b_2(S^*) = 3b_2(S)$, $\rho_r(S^*) \leqq 2 + \mathrm{b}_2(S^*)$. Therefore $3\rho_r(S) \leqq 2 + 3b_2(S)$ so that $\rho_r(S) \leqq b_2(S)$. Q.E.D.

(3.6) Let M be the maximal reduced divisor on S. Then we define a subset $N(S)$ of $H^1(S, \mathbf{C}^*)$ by

$$N(S) = \{F \in H^1(S, \mathbf{C}^*);\ h^0(S, \Omega_S^1(\log M)(F)) \neq 0\}$$

and $n(S)$ = the cardinality of $N(S)$.

(3.7) **Lemma.** *Assume $n(S)$ to be finite. Then $n(S) \leqq 2$.*

Proof. Assume that $3 \leqq n(S) < \infty$. Let F_ν be flat line bundles in $N(S)$, $F_\nu \neq F_\mu (\nu \neq \mu)$, ω_ν a global section of $\Omega_S^1(\log M)(F_\nu)$ $(\nu = 1,2,3)$. Assume $\omega_1 \wedge \omega_2 = 0$. Then we have at any point p of S a meromorphic function h_p such that $\omega_1 = h_p\omega_2$. Therefore we have a divisor D with $D = F_1 - F_2$. In view of (2.10) we have an effective divisor E whose associated line bundle is flat. Hence we have a nontrivial section $\omega_1 \cdot nE$ of $\Omega_S^1(\log M)(F_1 + nE)$, i.e. $F_1 + nE \in N(S)$ for $n > 0$. This implies that $n(S) = \infty$. Therefore $\omega_1 \wedge \omega_2 \neq 0$. Similarly $\omega_\nu \wedge \omega_\mu \neq 0\ (\nu \neq \mu)$. It follows that $h^0(S, \Omega_S^2(M)(F_\nu + F_\mu)) \neq 0$. Let $D_{\nu\mu} = (\omega_\nu \wedge \omega_\mu)$ be the divisor of $\omega_\nu \wedge \omega_\mu$. Then $D_{\nu\mu} = K + M + F_\nu + F_\mu$. Hence $D_{12} - D_{13} = F_2 - F_3$. In view of (2.10), we have a contradiction of $n(S)$ being finite in the same way above. Q.E.D.

(3.8) **Lemma.** *Assume that $n(S) = 2$ and let D be an effective divisor with $h^1(D, \mathcal{O}_D) = 2$ and $h^1(D', \mathcal{O}_{D'}) \leqq 1$ for any proper subcurve D' of D. Then D is two cycles of rational curves* (2.12.3) *and $F_1 + F_2 = 0$ for F_1, $F_2 \in N(S)$, $(F_1 \neq F_2)$. Moreover F_ν are of infinite order.*

Proof. Let F_1 and F_2 be line bundles in $N(S)$, $F_1 \neq F_2$. In view of (2.8) $K + D = 0$. By the proof of (3.7) there exists an effective divisor E such that $E = K + M + F_1 + F_2$, that is, $E + D = M + F_1 + F_2$. If $E + D \neq M$, then $n(S) = \infty$. Hence $E + D = M$ and $F_1 + F_2 = 0$. Since M is reduced, so is D. By (2.12) D is two cycles of rational curves. Assume that F_1 is of finite order. Let n be the order of F_1. Then we have an n-fold cyclic unramified covering $\pi: S^* \to S$ of S with covering group $\{\sigma^k; k = 0, \cdots, (n-1)\}$ such that $\pi^* F_1 = 0$ where $\sigma^n = id_{S^*}$. Then an element of $H^0(S, \Omega_S^1(\log M)(F_1))$ induces an element of $H^0(S^*, \Omega_{S^*}^1(\log \pi^* M))$. By (3.4) there exists an effective divisor H with $H^2 = 0$. By taking $\sum_{k=0}^{n-1} (\sigma^k)^* H$ instead of H we may assume that $\sigma^*(H) = H$ and $H^2 = 0$.

Then $\pi_*(H)$ is an effective divisor with $\pi_*(H)^2 = 0$. Therefore $[\pi_*(H)]$ is a flat line bundle and $n(S) = \infty$ which is a contradiction. Q.E.D.

(3.9) **Corollary.** $n(S) = \infty$ *if and only if there exists one of the curves* (2.10.1) $\sim$ (2.10.4).

(3.10) *Remark.* Let S be $S_\omega^{[n]}$. With the notations in (1.4) and (1.5) we define a character χ of G_n by $\chi(g_0^n) = \alpha_0^n$. Then by the isomorphism of $H^1(S, \mathbf{C}^*)$ with $\mathrm{Hom}(H_1(S, \mathbf{Z}), \mathbf{C}^*) \cong \mathrm{Hom}(G_n, \mathbf{C}^*)$, we have a flat line bundle F_χ on S associated with χ. By [6] p. 100 $S - A - B$ is a quotient space of $\mathbf{H} \times \mathbf{C}$ by the action of $\mathbf{Z} + \mathbf{Z}\omega$ and G_n, it is easy to check that $d\xi$ and $d\zeta$ can be extended to sections of $\Omega^1_{\mathscr{D}}(\log(C+D))$ by

$$d\xi = \omega x_0^{-1} dx_0 + y_0^{-1} dy_0, \quad d\zeta = \omega' x_0^{-1} dx_0 + y_0^{-1} dy_0$$

and they generate subsheaves of $\Omega^1_S(\log(A+B))$, isomorphic to $\mathcal{O}_S(F_\chi)$ and $\mathcal{O}_S(-F_\chi)$ respectively. In fact, we have by this

$$\Omega^1_S(\log(A+B)) \cong \mathcal{O}_S(F_\chi) \oplus \mathcal{O}_S(-F_\chi).$$

Therefore $\Omega^1_S(\log(\mathrm{A}+\mathrm{B}))_A \cong \mathcal{O}_A(F_\chi) \oplus \mathcal{O}_A(-F_\chi)$, $\Omega^1_S(\log(\mathrm{A}+\mathrm{B}))_B \cong \mathcal{O}_B(F_\chi) \oplus \mathcal{O}_B(-F_\chi)$.

(3.11) **Lemma.** *Let A and B be cycles of rational curves on S and assume that b_2 = the number of irreducible components of $A+B$. Then we have $n(S) = 2$.*

Proof. By (2.2.1) (2.12) (3.7) and (3.9) we have $n(S) \leqq 2$. In view of (2.8) $K + A + B = 0$. In view of (2.9), we have the maximal reduced divisor $M = A + B$. Since the analytic structure of a neighborhood of A (or B) in S is uniquely determined by the sequence of self-intersection numbers of irreducible components of A (or B respectively) ([15]), we have by (3.10) a flat line bundle $\bar{F}$ on A (or $\bar{G}$ on B) of infinite order such that

$$\Omega^1_S(\log M)_A = \mathcal{O}_A(\bar{F}) \oplus \mathcal{O}_A(-\bar{F}),$$
$$\Omega^1_S(\log M)_B = \mathcal{O}_B(\bar{G}) \oplus \mathcal{O}_B(-\bar{G}).$$

By virture of (2.14) and (2.13.1) there exist flat line bundles F and G on S such that $F_A = \bar{F}$ and $G_B = \bar{G}$.

For brevity we denote $\Omega^1_S(\log M)$ by $\tilde{\Omega}_S$. We shall prove either $h^0(S, \tilde{\Omega}_S(F)) \neq 0$ or $h^0(S, \tilde{\Omega}_S(-F)) \neq 0$. We assume the contrary. The following is exact;

$$\begin{aligned} 0 &\to H^0(S, \tilde{\Omega}_S(-M+F)) \to H^0(S, \tilde{\Omega}_S(F)) \to H^0(M, \tilde{\Omega}_S(F)_M) \\ &\to H^1(S, \tilde{\Omega}_S(-M+F)) \to H^1(S, \tilde{\Omega}_S(F)) \to H^1(M, \tilde{\Omega}_S(F)_M) \\ &\to H^2(S, \tilde{\Omega}_S(-M+F)) \to H^2(S, \tilde{\Omega}_S(F)) \to 0. \end{aligned}$$

By assumption $h^0(S, \tilde{\Omega}_S(F)) = 0$ so that $h^0(S, \tilde{\Omega}_S(-M+F)) = 0$. Since Ω_S is self-dual, it follows that $h^2(S, \tilde{\Omega}_S(-M+F)) = h^0(S, \tilde{\Omega}_S(-F)) = 0$. On the other hand we have

$$\begin{aligned}\chi(S,\tilde{\Omega}_S(-M+F)) &= \chi(S,\tilde{\Omega}_S(F)) - \chi(M,\tilde{\Omega}_S(F)_M)\\ &= \chi(S,\tilde{\Omega}_S) - \chi(M,(\tilde{\Omega}_S)_M)\\ &= \chi(S,\Omega^1_S) + \chi(\tilde{M},\mathcal{O}_{\tilde{M}}) - \chi(M,(\tilde{\Omega}_S)_M)\\ &= -b_2 + h^0(\tilde{M},\mathcal{O}_{\tilde{M}}) \quad \text{(by the proof of (3.3))}\\ &= 0 \quad \text{(by assumption)}.\end{aligned}$$

Therefore we have $h^1(S,\tilde{\Omega}_S(-M+F)) = 0$. This contradicts that $h^0(M,\tilde{\Omega}_S(F)_M) \geqq h^0(A,\mathcal{O}_A) = 1$. So we may assume $h^0(S,\tilde{\Omega}_S(F)) \neq 0$. Similarly we may assume $h^0(S,\tilde{\Omega}_S(G)) \neq 0$. If $F \neq G$, then $n(S) = 2$ and $F + G = 0$ by virtue of (3.8). So (3.11) follows in this case. So we assume that $F = G$. Assume $h^0(S,\tilde{\Omega}_S(-F)) = 0$ to derive a contradiction. From the exact sequence

$$\begin{aligned}0 &\to H^0(S,\tilde{\Omega}_S(-M-F)) \to H^0(S,\tilde{\Omega}_S(-F)) \to H^0(M,\tilde{\Omega}_S(-F)_M)\\ &\to H^1(S,\tilde{\Omega}_S(-M-F)) \to H^1(S,\tilde{\Omega}_S(-F)) \to H^1(M,\tilde{\Omega}_S(-F)_M)\\ &\to H^2(S,\tilde{\Omega}_S(-M-F)) \to H^2(S,\tilde{\Omega}_S(-F)) \to 0,\end{aligned}$$

we infer $h^1(S,\tilde{\Omega}_S(-M-F)) \geqq 2$. In view of Riemann Roch and Serre duality, we have $h^1(S,\tilde{\Omega}_S(-M-F)) = h^0(S,\tilde{\Omega}_S(-M-F)) + h^2(S,\tilde{\Omega}_S(-M-F)) = h^0(S,\tilde{\Omega}_S(F))$. Since F is of infinite order, in particular $2F \neq 0$, we have $h^0(S,\tilde{\Omega}_S(F)) = 1$ by the same argument as in (3.8). This is a contradiction. Therefore $h^0(S,\tilde{\Omega}_S(-F)) \neq 0$ i.e. $-F \in N(S)$. Since $F \neq -F$, we have $n(S) = 2$ by (3.7). Q.E.D.

(3.12) **Lemma.** *Let C be a cycle of rational curves on S with $C^2 < 0$. Assume that b_2 = the number of irreducible components of C. Then $n(S) = 2$ and we have a flat line bundle L on S and an unramified double covering $\pi\colon S^* \to S$ of S such that*

$$L_C = 0, \quad 2L = 0, \quad \pi^* L = 0, \quad F_1 + F_2 = L \neq 0$$

for F_1, $F_2 \in N(S)$.

Proof. Let $C = \sum_{\nu=1}^{r} C_\nu$ be the decomposition of C into irreducible components. Since $C^2 < 0$ and $b_2 = r$, the topological two cycles $C_\nu (\nu = 1, \cdots, r)$ form a basis of $H_2(S,\mathbf{R})$. Since $KC_\nu = -2 - C_\nu^2 = -CC_\nu$ for any ν, we have $K + C = 0$ in $H^2(S,\mathbf{Q})$ so that there exists a flat line bundle L on S such that $K + C = L$. If $L = 0$, then $h^1(C,\mathcal{O}_C) = 2$ follows from (2.6) and $h^2(S,\mathcal{O}_S(-C)) = 1$, which is a contradiction. Hence $L \neq 0$. The dualising sheaf ω_C of C is trivial so that $L_C = (K + C)_C = \omega_C = 0$. This implies by the proof of (2.13) that $2L = 0$ and that there exists an unramified double covering $\pi\colon S^* \to S$ such that $\pi^* L = 0$ and $\pi^{-1}(C)$ is two cycles of rational curves, $b_2(S^*) = 2r$. Clearly C and $\pi^* C$ are the maximal effective reduced divisors of S and S^* respectively. By (3.10) we have a flat line bundle G on S^* with $h^0(S^*,\tilde{\Omega}_{S^*}(\pm G)) = 1$. Moreover from (2.13.1) and (2.14) we infer a commutative diagram with exact rows;

$$\begin{tikzcd} 1 \arrow[r] & 1 \arrow[r] & H^1(S^*,\mathbf{C}^*) \arrow[r, "j^*"] & H^1(C',\mathbf{C}^*) \arrow[r] & 1 \\ 1 \arrow[r] & \{1,L\} \arrow[r] \arrow[u, "\pi^*"] & H^1(S,\mathbf{C}^*) \arrow[r, "i^*"] \arrow[u, "\pi^*"] & H^1(C,\mathbf{C}^*) \arrow[r] \arrow[u, "\pi^*"] & 1 \end{tikzcd}$$

where C' is a connected component of $\pi^{-1}(C)$, i and j are inclusion mappings of C and C' into S and S^* respectively. We notice that C and C' are isomorphic and therefore $\pi^*\colon H^1(C, \mathbf{C}^*) \to H^1(C', \mathbf{C}^*)$ is an isomorphism.

As $\pi^*\colon H^1(S, \mathbf{C}^*) \to H^1(S^*, \mathbf{C}^*)$ is therefore surjective, we have a flat line bundle F on S such that $\pi^* F = G$. Since $\pm \pi^* F \in N(S^*)$ and

$$\pi_* \pi^* \tilde{\Omega}_S = \tilde{\Omega}_S \otimes \pi_* \mathcal{O}_{S^*} = \tilde{\Omega}_S \oplus \tilde{\Omega}_S \otimes L,$$

we have

$$h^0(S, \tilde{\Omega}_S(F)) + h^0(S, \tilde{\Omega}_S(F+L)) = 1$$
$$h^0(S, \tilde{\Omega}_S(-F)) + h^0(S, \tilde{\Omega}_S(-F+L)) = 1\,.$$

Since $\pi^*(F+L) = G$, we may assume that $h^0(S, \tilde{\Omega}_S(F)) = 1$, i.e. $F \in N(S)$. Then by $K_S + C = L$, we have $h^0(S, \tilde{\Omega}_S(-F+L)) = 1$. In view of (3.7) and (3.9) we have $n(S) = 2$. Q.E.D.

§4. Deformations of neighborhoods of cycles

(4.1) Let S be a surface, A a cycle of rational curves on S with negative definite intersection matrix. Suppose that A contains no exceptional curves of the first kind. Then the cycle A can be contracted to a normal point by [3], as is the same, we can choose a strictly pseudoconvex open neighbood U of A in S. For any coherent sheaf $\mathscr{F}$ on U, $H^q(U, \mathscr{F})$ is finite-dimensional for $q = 1$, zero for $q = 2$.

(4.2) *Definitions.* $\Theta_S(-\log A) := \mathscr{H}om_{\mathcal{O}_S}(\Omega_S^1(\log A), \mathcal{O}_S)$,

$$J_A := \Theta_S / \Theta_S(-\log A)\,.$$

(4.3) **Lemma.** $H^1(U, \Theta_U) \cong H^1(A, (\Theta_U)_A) \cong H^1(A, J_A)$.

Proof. We have an exact sequence;

$$0 \to H^0(U, \Theta_U(-\log A)) \to H^0(U, \Theta_U) \to H^0(A, J_A)$$
$$\to H^1(U, \Theta_U(-\log A)) \to H^1(U, \Theta_U) \to H^1(A, J_A) \to 0\,.$$

Since the analytic structure of a sufficiently small neighborhood of the cycle A is uniquely determined by the sequence of self-intersection numbers of irreducible components of A ([15, p. 139]), U is isomorphic to an open neighborhood of one of the cycles on a hyperbolic Inoue surface $S_\omega^{[n]}$. Hence by (3.10) and $\Omega_U^2 \cong \mathcal{O}_U(-A)$, we have $\Theta_U(-\log A) \cong \Omega_U^1(\log A) \cong \mathcal{O}_U(F) \oplus \mathcal{O}_U(-F)$ for a flat line bundle F with F_A of infinite order. By [14, Theorem 5.4] we have $H^q(U, F) \cong H^q(A, F_{nA})$ for any q and any sufficiently large n. We shall show by the induction on n that $H^1(A, G_{nA}) = 0$ for $n > 0$ and any flat line bundle G on U such that G_A is nontrivial. Let $n = 1$. Then $H^1(A, G_A) = 0$ because $G_A \neq 0$ and $H^0(A, G_A) = 0$. We assume next $H^1(A, G_{nA}) = 0$. From an exact sequence

$$0 \to \mathcal{O}_A(-nA + G) \to \mathcal{O}_{(n+1)A}(G) \to \mathcal{O}_{nA}(G) \to 0\,,$$

we infer an exact sequence

$$0 \to H^0(A, \mathcal{O}_A(-nA + G)) \to H^0(A, G_{(n+1)A}) \to H^0(A, G_{nA})$$
$$\to H^1(A, \mathcal{O}_A(-nA + G)) \to H^1(A, G_{(n+1)A}) \to H^1(A, G_{nA}) \to 0\,.$$

Then $h^1(A,\mathcal{O}_A(-nA+G)) = h^0(A,\mathcal{O}_A(nA-G))=0$ since $A_\lambda(nA-G)\leqq 0$ for any irreducible component A_λ of A and $A_\nu(nA-G)<0$ for at least one A_ν.

Hence $H^1(A,G_{(n+1)A}) = H^1(A,G_{nA})=0$. Hence $H^1(U,\Theta_U(-\log A)) = \varprojlim (H^1(A,F_{nA})\otimes H^1(A,-F_{nA}))=0$, $H^1(U,\Theta_U)\cong H^1(A,J_A)$. Next we shall show $H^1(A,(\Theta_S)_A)\cong H^1(A,J_A)$. By a direct computation the following is exact; $0\to\mathcal{O}_{\tilde{A}}\to(\Theta_S)_A\to J_A\to 0$ where $\tilde{A}$ is the normalisation of A. Clearly $H^1(\tilde{A},\mathcal{O}_{\tilde{A}}) = 0$, hence $H^1(A,(\Theta_S)_A)\cong H^1(A,J_A)$. Q.E.D.

(4.4) Let S be a VII_0 surface with an elliptic curve E and a cycle Z of rational curves. Then by (2.8) $K+E+Z=0$. By (2.12) $Z^2=0$, $EZ=0$ and $E^2=c_1^2=-b_2$. Now let p be a nonsingular point of Z, and $h\colon S'\to S$ be the blowing-up of S at p, E' and Z' the proper transforms of E and Z. Then we have $(Z')^2=-1$ so that the intersection matrix of Z' is negative definite. Let U' be a strictly pseudoconvex neighborhood of Z'. By (4.3) $H^1(U',\Theta_{U'})\cong H^1(Z',J_{Z'})$. On the other hand we have the following exact sequence;

$$\begin{aligned}0&\to H^0(S',\Theta_{S'}(-\log(E'+Z')))\to H^0(S',\Theta_{S'}(-\log E'))\to H^0(Z',J_{Z'})\\&\to H^1(S',\Theta_{S'}(-\log(E'+Z')))\to H^1(S',\Theta_{S'}(-\log E'))\to H^1(Z',J_{Z'})\\&\to H^2(S',\Theta_{S'}(-\log(E'+Z')))\to H^2(S',\Theta_{S'}(-\log E'))\to 0.\end{aligned}$$

Therefore

$$\begin{aligned}h^2(S',\Theta_{S'}(-\log(E'+Z'))) &= h^0(S',\Omega^1_{S'}(\log(E'+Z'))(K_{S'}))\\&= h^0(S',\Omega^1_{S'}(\log(E'+Z'))\,(-E'-Z'))\leqq h^0(S',\Omega^1_{S'})=0,\end{aligned}$$

a fortiori

$$h^2(S',\Theta_{S'}(-\log(E'+Z'))) = h^2(S',\Theta_{S'}(-\log E'))=0.$$

Since

$$J_{Z'}\cong\mathcal{O}_{\mathbf{P}_1}(-3)\oplus\bigoplus_{a=1}^{n-1}\mathcal{O}_{\mathbf{P}_1}(-2),$$

we have

$$\begin{aligned}H^0(S',\Theta_{S'}(-\log E')) &= H^0(S',\Theta_{S'}(-\log(E'+Z')))\\&\cong H^0(S',\Omega^1_{S'}(\log(E'+Z')))=0\end{aligned}$$

because $h^0(S',\Omega^1_{S'}(\log(E'+Z')))$ = the number of divisors (2.10.1) ~ (2.10.4) with support in $E'+Z'$ by the same argument as in (3.4). Thus, combining the above with (4.3), we have,

(4.5) **Lemma.** $H^q(S',\Theta_{S'}(-\log E'))=0$ $(q=0,2)$ *and the following sequence is exact,*

$$0\to H^1(S',\Theta_{S'}(-\log(E'+Z')))\to H^1(S',\Theta_{S'}(-\log E'))\to H^1(U',\Theta_{U'})\to 0.$$

(4.6) **Lemma.** *Let S be a* VII_0 *surface with an elliptic curve E and a cycle Z of rational curves. Then there exists a smooth family $\pi\colon\mathcal{S}\to D$ with two divisors E and Z flat over D such that $(\mathcal{S},\mathcal{E},\mathcal{Z})_0=(S,E,Z)$, $\mathcal{E}_t=E$ and $\mathcal{Z}_t$ is a smooth elliptic curve for any $t\neq 0$ where D is the unit disc.*

Proof. With the notations of (4.4), there exists by [10, Corollary 4] a semi-universal deformation $(\mathcal{S}'-\mathcal{E}', \mathcal{S}', \mathcal{E}', f, T', 0)$ of logarithmic deformations of $(S'-E', S', E')$ such that the Kodaira-Spencer mapping $\rho_0: T_{T',0} \to H^1(S', \Theta_{S'}(-\log E'))$ is an isomorphism. This induces an epimorphism of $T_{T',0}$ onto $H^1(U', \Theta_{U'})$ for a strictly pseudoconvex neighborhood U' of Z' by (4.5). Hence by [16, Theorem 5] there exists an open neighborhood $\mathcal{U}'$ of Z' in $\mathcal{S}'$ such that $\mathcal{S}'_0 \cap \mathcal{U}' \cong U'$, $f_{\mathcal{U}'}: \mathcal{U}' \to T'$ is versal in the sense of [16, Theorem 5]. On the other hand by [7] there is a one parameter family $g: \mathcal{U}'' \to D$ of deformations of U' and a divisor $\mathcal{Z}''$ of $\mathcal{U}''$ flat over D such that $\mathcal{Z}''_0 \cong Z'$, $\mathcal{Z}''_t$ is a smooth elliptic curve for any $t \neq 0$. By the versality of $f_{\mathcal{U}'}$, we have a one-parameter smooth family $\pi': \mathcal{S} \to D$, and an open neighborhood $\mathcal{V}$ of Z' in $\mathcal{S}'$ such that $\mathcal{S}'_0 \cong S'$, $\mathcal{V} \cong \mathcal{U}''$. Hence we have a divisor $\mathcal{W}$ of $\mathcal{S}'$ flat over D such that $\mathcal{W} \cong \mathcal{Z}''$. By stability of the exceptional curve of the first kind [13, Theorem 5], we have a divisor $\mathcal{C}$ of $\mathcal{S}'$ flat over D such that $\mathcal{C}_0 = h^{-1}(p)$ in (4.4) and $\mathcal{C}$ can be blown down simultaneously so that we have a smooth family $\pi: \mathcal{S} \to D$ over the disc D with two divisors $\mathcal{E}$ and $\mathcal{Z}$ flat over D such that $(\mathcal{S}, \mathcal{E}, \mathcal{Z})_0 \cong (S, E, Z)$, $\mathcal{E}_t \cong E$ and $\mathcal{Z}_t$ is a smooth elliptic curve for any $t \neq 0$. Q.E.D.

(4.7) **Lemma.** *Let S be a* VII$_0$ *surface with A and B cycles of rational curves. Then we can choose strictly pseudoconvex open neighborhoods U and V of A and B in S respectively and*

(4.7.1) $H^q(S, \Theta_S) = H^q(\Theta_S(-\log A)) = H^q(S, \Theta_S(-\log B))$
$= H^q(S, \Theta_S(-\log(A+B)) = 0 \quad (q = 0, 2)$,

(4.7.2) *the following sequences are exact*;

$$0 \to H^1(S, \Theta_S(-\log(A+B))) \to H^1(S, \Theta_S) \to H^1(U, \Theta_U) \oplus H^1(V, \Theta_V) \to 0,$$
$$0 \to H^1(S, \Theta_S(-\log(A+B))) \to H^1(S, \Theta_S(-\log A)) \to H^1(V, \Theta_V) \to 0$$
$$0 \to H^1(S, \Theta_S(-\log(A+B))) \to H^1(S, \Theta_S(-\log B)) \to H^1(U, \Theta_U) \to 0.$$

Proof. By (2.12) $A^2 < 0$ and $B^2 < 0$, hence the intersection matrices of A and B are negative definite. Therefore we can choose strictly pseudoconvex open neighborhoods U and V of A and B respectively. By (4.3) $H^1(U, \Theta_U) \cong H^1(A, J_A)$, $H^1(V, \Theta_V) \cong H^1(B, J_B)$. Let $A = \sum_{\lambda=1}^{r} A_\lambda$, $B = \sum_{\nu=1}^{s} B_\nu$ be the decompositions of A and B into irreducible components. Then $J_A \cong \bigoplus_{\lambda=1}^{r} \mathcal{O}_{\mathbf{P}_1}(A_\lambda^2)$, $J_B \cong \bigoplus_{\nu=1}^{r} \mathcal{O}_{\mathbf{P}_1}(B_\nu^2)$. Therefore $H^0(A, J_A) = H^0(B, J_B) = 0$. It follows from this in the same manner as in (4.4) that $h^0(S, \Theta_S) = h^0(S, \Theta_S(-\log A)) = h^0(S, \Theta_S(-\log B)) = h^0(S, \Theta_S(-\log(A+B))) =$ the number of curves (2.10.1) ~ (2.10.4) with supports in $A + B = 0$. On the other hand $h^2(S, \Theta_S(-\log(A+B))) = h^0(S, \Omega_S^1(\log(A+B))(K_S)) = 0$ because $K_S = -A - B$ by (2.8). Hence $h^2(S, \Theta_S) = h^2(S, \Theta_S(-\log A)) = h^2(S, \Theta_S(-\log B)) = 0$. Now (4.7.2) is clear. Q.E.D.

(4.8) **Lemma.** *Let S be a* VII$_0$ *surface with two cycles A and B of rational curves. Then there exists a smooth proper family $\pi: \mathcal{S} \to D$ with two divisors $\mathcal{A}$ and $\mathcal{B}$ flat over D such that $(\mathcal{S}, \mathcal{A}, \mathcal{B})_0 \cong (S, A, B)$, $\mathcal{A}_t$ is a smooth elliptic curve $(t \neq 0)$ and $\mathcal{B}_t = B$ for any t.*

Proof. By (4.7.1) and [10, Corollary 4] there exists a semiuniversal deformation $(\mathcal{S}^* - \mathcal{B}^*, \mathcal{S}^*, \mathcal{B}^*, f, T, 0)$ of logarithmic deformations of $(S - B, S, B)$ such that $(\mathcal{S}^*, \mathcal{B}^*)_0 = (S, B)$, the Kodaira-Spencer mapping $\rho_0 \colon T_{T,0} \to H^1(S, \Theta_S(-\log B))$ is an isomorphism. Then by (4.7.2) and the same argument as in (4.6) we can show that there exists a one-parameter smooth proper family $\pi \colon \mathcal{S} \to D$ with two divisors $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}_0 = A$, $\mathcal{A}_t$ is a smooth elliptic curve $(t \neq 0)$ and $\mathcal{B}_t \cong B$ for any t. Q.E.D.

(4.9) **Lemma.** *Let S be a* VII_0 *surface with two cycles A and B of rational curves. Then there exists a one-parameter smooth proper family π; $\mathcal{S} \to D$ with two divisors $\mathcal{A}$ and $\mathcal{B}$ flat over D such that $(\mathcal{S}, \mathcal{A}, \mathcal{B})_0 = (S, A, B)$, $\mathcal{A}_t$ and $\mathcal{B}_t$ are smooth elliptic curves for any $t \neq 0$.*

Proof. Similar to (4.8). Q.E.D.

§5. Hopf surfaces

(5.1) *Definition.* A surface S is called a diagonal Hopf surface if S is isomorphic to a quotient $\mathbf{C}^2 - (0,0)/\{g^n; n \in \mathbf{Z}\}$ where g is a transformation of $\mathbf{C}^2 - (0,0) : (z_1, z_2) \to (\alpha_1 z_1, \alpha_2 z_2)$ with $0 < |\alpha_i| < 1$. A surface is called a Hopf surface if the universal covering of it is $\mathbf{C}^2 - (0,0)$. A Hopf surface is called primary if its fundamental group is infinite cyclic.

(5.2) **Theorem** (Kato). *Let S be a* VII_0 *surface with two elliptic curves. Assume that S has no meromorphic functions except constants. Then S is isomorphic to a quotient surface of* $\mathbf{C}^2 - (0,0)$ *by the group generated by two transformations g and h*

$$g \colon (z_1, z_2) \to (\alpha_1 z_1, \alpha_2 z_2),$$
$$h \colon (z_1, z_2) \to (\exp(2\pi i/n) z_1, \exp(2\pi i p/n) z_2)$$

where $0 < |\alpha_i| < 1$, *n and p are relatively prime positive integers. If moreover* $H_1(S, \mathbf{Z}) \cong \mathbf{Z}$, *then S is a diagonal Hopf surface.*

Our proof is a slight modification of Kato's original proof (unpublished).

Proof. Let A and B be elliptic curves on S. By (2.12) we have $A^2 = B^2 = 0$, $K_S = -A - B$, hence $b_2 = -c_1^2 = -(A^2 + B^2) = 0$. Then our theorem follows from [12, II, Theorems 32 and 34]. Q.E.D.

(5.3) **Lemma.** *Let S be a* VII_0 *surface with two elliptic curves. Suppose that* $H_1(S, \mathbf{Z}) \cong \mathbf{Z}$, *and* $h^0(S, mK_S) = 0$ *for any* $m > 0$. *(S may have nonconstant meromorphic functions.) Suppose moreover that any unramified Galois covering S^* of S is cyclic and* $H_1(S^*, \mathbf{Z}) \cong \mathbf{Z}$. *Then S is isomorphic to a diagonal Hopf surface.*

Proof. Assume first that S has no meromorphic functions except constants. Then by (5.2) S is isomorphic to a diagonal Hopf surface. Next we assume that S has nonconstant meromorphic functions. Then S is an elliptic surface by [11, I, Theorem 4.1]. Let $f \colon S \to \Delta$ be the elliptic fibration. By [12, II, Theorem 27] S is isomorphic to an elliptic surface

$$L_{a_r}(m_r, \beta_r) \cdots L_{a_1}(m_1, \beta_1)(\Delta \times C)$$

where Δ is the projective line and C is a nonsingular elliptic curve, $a_\lambda \in \Delta$, $a_\lambda \neq a_\nu (\lambda \neq \nu)$, $[\beta_\lambda]$ is a point of C of order m_λ $\beta_1 + \cdots + \beta_r \neq 0$. Following [12, II] we define an invariant κ by $\kappa = -2 + \sum_{\nu=1}^{r} (1 - (1/m_\nu))$. Let $C_u = f^{-1}(u)$, $P_\nu = (C_{a_\nu})_{\mathrm{red}}$. Since $K_S = -2C_u + \sum_{\nu=1}^{r} (m_\nu - 1) P_\nu$, we infer $\kappa < 0$, $1 \leqq r \leqq 3$ from $h^0(S, mK_S) = 0$.

Case 1. Assume that $r = 3$. Take a simply connected Galois covering $\tilde{\omega} : \Gamma \to \Delta$ which is unramified over $\Delta - \{a_1, a_2, a_3\}$ and has branch points of order $m_\nu - 1$ over a_ν. In fact, since the possible triples (m_1, m_2, m_3) are $(2, 2, n)$ $(n \geqq 2)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, we can take respectively the dihedral groups, the tetrahedral group, the octahedral group and the icosahedral group as the covering group G, and the projective line as Γ. Let S^* be the elliptic surface induced from $f: S \to \Delta$ by $\tilde{\omega}$. Then S^* is an unramified Galois covering of S with covering group G, however by the assumption the covering group should be cyclic which is a contradiction. Hence $r \neq 3$.

Case 2. Assume $r = 2$. Letting $a_1 = 0$, $a_2 = \infty$ we have $S = L_0(m_1, \beta_1) L_\infty(m_2, \beta_2) (\Delta \times C)$, $[\beta_j]$ being points of order m_j of C with $\beta_1 + \beta_2 \neq 0$ in $\mathbf{C}$ and $\Delta = \mathbf{P}_1$. Let $d = (m_1, m_2)$ (:= the greatest common divisor of m_1 and m_2), $e_j = m_j/d$. Let $S^* = L_0(e_1, d\beta_1)\ L_\infty(e_2, d\beta_2) (\Delta \times C)$. Then by [12, II. p. 688] S^* is an unramified covering of S, hence by assumption $H_1(S^*, \mathbf{Z}) \cong \mathbf{Z}$. By multiplying the global coordinate of C by a suitable constant we may assume $e_1 e_2 d(\beta_1 + \beta_2) = n$ (a positive integer) and the lattice of C is $\mathbf{Z} + \mathbf{Z}\omega$. Now that e_1 and e_2 are relatively prime, $H_1(S^*, \mathbf{Z}) \cong \mathbf{Z} + (\mathbf{Z}/n\mathbf{Z})$ by [ibid., p. 688]. Hence $n = 1$. By [ibid., p. 688] S^* is written as $S^* = (\mathbf{C} \times C)/G_1 \cup (\mathbf{C} \times C)/G_2$ where $((u_1, \zeta_1))$ and $((u_2, \zeta_2))$ are identical if and only if $u_1^{e_1} u_2^{e_2} = 1$, $\zeta_1 + (m_1 \beta_1 \log u_1)/2\pi i = \zeta_2 + (m_2 \beta_2 \log u_2)/2\pi i$ $\mathrm{mod}\, \mathbf{Z} + \mathbf{Z}\omega$ and the group G_j is generated by a transformation g_j: $(u_j, \zeta_j) \to (\exp(2\pi i/e_j) u_j, \zeta_j - d\beta_j)$. By [ibid.] we have a holomorphic mapping f of a diagonal Hopf surface onto S^* as follows. Let h be a transformation of $\mathbf{C}^2 - (0,0)$, $h: (z_1, z_2) \to (\exp(2\pi i e_1 \omega) z_1, \exp(2\pi i e_2 \omega) z_2)$. Then we have a diagonal Hopf surface S_0 as the quotient of $\mathbf{C}^2 - (0,0)$ by the infinite cyclic group generated by h. We define a mapping f of S_0 onto S^* by

$$f^*(u_1, \zeta_1) = (z_2 \exp(-(e_2/e_1) \log z_1), (\log z_1)/2\pi i e_1),$$

$$f^*(u_2, \zeta_2) = (z_1 \exp(-(e_1/e_2) \log z_2), (\log z_2)/2\pi i e_2).$$

Then f is in fact an isomorphism by [ibid.]. This shows that S^* is a diagonal Hopf surface. Therefore S is a quotient of S_0 by an automorphism $q: (z_1, z_2) \to (\exp(-2\pi i e_1 \beta_1) z_1, \exp(2\pi i e_2 \beta_2) z_2)$. Hence $\pi_1(S)$ is abelian, so it is infinite cyclic by the assumption. Any generator of $\pi_1(S)$ is a product of some powers of h and q. So S is a diagonal Hopf surface.

Case 3. Assume $r = 1$. Then $S = L_\infty(m, \beta) (\mathbf{P}_1 \times C)$. We may assume that $m\beta = n$ (a positive integer) and the lattice of $C = \mathbf{Z} + \mathbf{Z}\omega$ by multiplying the global coordinate of C by a suitable constant. By [ibid., p. 688] $S = \mathbf{C} \times C/G \cup \mathbf{C} \times C$ where $((s, \zeta))$ and (t, ξ) are identical if and only if $t^{-1} = s^m$, $\xi = \zeta + (\log s)/2\pi i$ $\mathrm{mod}\, \mathbf{Z} + \mathbf{Z}\omega$ and G is the group generated by a transformation $g: (s, \zeta) \to (\exp(2\pi i/m) s, \zeta - \beta)$. Since $[\beta]$ is a point of order m, the integers m and n are relatively prime. Hence there exist integers p and q such that $pm+qn=1$. The group G is generated by g^q. In fact $g=(g^q)^n$. Therefore by taking g^q instead of g, we may assume $\beta = 1/m$. We define a holomorphic mapping f of $\mathbf{C}^2-(0,0)$ into S by

$$f^*(s,\zeta) = (z_2 \exp(-(1/m)\log z_1),\ \log z_1/2\pi i m),$$
$$f^*(t,\xi) = (z_1 z_2^{-m},\ \log z_2/2\pi i).$$

Let h be a transformation of $\mathbf{C}^2-(0,0)$: $(z_1,z_2) \to (\exp(2\pi i m\omega) z_1, \exp(2\pi i\omega) z_2)$, H a group generated by h. Then H operates on $\mathbf{C}^2-(0,0)$ properly discontinuously and freely. The quotient $S_0 := \mathbf{C}^2-(0,0)/H$ is a compact complex surface and turns out to be isomorphic to S by f. Hence S is a diagonal Hopf surface. Q.E.D.

(5.4) **Lemma.** *Let S be a* VII$_0$ *surface with two cycles A and B of rational curves. Let π: $\mathcal{S} \to D$ be a one-parameter smooth proper family, $\mathcal{A}$ and $\mathcal{B}$ divisors of S flat over D such that $(\mathcal{S},\mathcal{A},\mathcal{B})_0 \cong (S,A,B)$ and $\mathcal{A}_t$ and $\mathcal{B}_t$ $(t \neq 0)$ are smooth elliptic curves. Then for any small $|t|$ $(t \neq 0)$, $\mathcal{S}_t$ is a diagonal Hopf surface with two elliptic curves $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{B}}_t$ blown up with centers points lying on $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{B}}_t$. The curves $\mathcal{A}_t$ and $\mathcal{B}_t$ are proper transforms of $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{B}}_t$.*

Proof. We have $H^0(S, mK_S)=0$ for any $m>0$ by (2.8). For any small $|t|$, $H^0(\mathcal{S}_t, mK_{\mathcal{S}_t})=0$ by the upper-semicontinuity. By (2.14) $H_1(\mathcal{S}_t,\mathbf{Z}) \cong H_1(S,\mathbf{Z}) \cong \mathbf{Z}$. Let $\mathcal{X}_t$ be the minimal model of S_t. Then we have $h^0(\mathcal{X}_t, mK_{\mathcal{X}_t})=0$, $H_1(\mathcal{X}_t,\mathbf{Z}) \cong \mathbf{Z}$ and there are two elliptic curves $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{B}}_t$ on $\mathcal{X}_t$ that are images of $\mathcal{A}_t$ and $\mathcal{B}_t$. Since $\pi_1(\mathcal{X}_t) \cong \pi_1(\mathcal{S}_t) \cong \pi_1(S)$, there exists an unramified Galois covering f: $S^* \to S$ with covering group G for any unramified Galois covering h: $X^* \to \mathcal{X}_t$ with covering group G. As there are at most two cycles of rational curves on S^*, hence the inverse images of A and B only, the covering f must be cyclic and by (2.14), $H_1(S^*,\mathbf{Z}) \cong \mathbf{Z}$. Hence G is cyclic and $H_1(X^*,\mathbf{Z}) \cong \mathbf{Z}$. Consequently by (5.3) $\mathcal{X}_t$ is a diagonal Hopf surface with two elliptic curves $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{B}}_t$ and therefore $\bar{\mathcal{A}}_t^2 = \bar{\mathcal{B}}_t^2 = 0$. So $b_2(\mathcal{S}_t)$ = the times of blowing-ups of $\mathcal{X}_t$. Since $c_1(\mathcal{S}_t)^2 = c_1(S)^2 = -A^2-B^2 = -\mathcal{A}_t^2 - \mathcal{B}_t^2$, and $c_1(\mathcal{S}_t)^2 = -b_2(\mathcal{S}_t)$, the centers of blowing-ups of $\mathcal{X}_t$ lie on the curves (or the proper transforms of) $\bar{\mathcal{A}}_t$ and $\bar{\mathcal{B}}_t$. Q.E.D.

(5.5) Let $D=\{t \in \mathbf{C}; |t|<1\}$, $D'=D-\{0\}$, $T=D' \times D'$. Define a transformation g of $T \times (\mathbf{C}^2-(0,0))$ by $g(t_1,t_2,z_1,z_2)=(t_1,t_2,t_1z_1,t_2z_2)$ and let G be the group generated by g. Then it is easy to see that the group G acts on $T \times (\mathbf{C}^2-(0,0))$ properly discontinuously without fixed points. Let $\mathcal{X}$ be the quotient of $T \times (\mathbf{C}^2-(0,0))$ by G, π: $\mathcal{X} \to T$ the natural projection. Then π is proper and smooth, any fiber $\pi^{-1}(t)$ is a diagonal Hopf surface. For a pair of holomorphic functions α_1 and α_2 with $0<|\alpha_\nu(t)|<1$ $(t \in D', \nu=1,2)$, we define $\mathcal{X}(\alpha_1,\alpha_2)$ to be the pull back of $\mathcal{X}$ by the holomorphic mapping (α_1,α_2) of D' into T.

(5.6) **Lemma.** *By choosing a general family π: $\mathcal{S} \to D$ in* (5.4), *there exists a dense subset D^* of D such that $\mathcal{S}_t$ has no meromorphic functions except constants for $t \in D^*$.*

In order to prove (5.6) we suffice to prove the following two lemmas. See (5.9).

(5.7) **Lemma.** *Let S be a* VII$_0$ *surface with two cycles A and B of rational curves. Let $\tilde{\omega}$: $\mathcal{S}' \to T'$ be a semiuniversal deformation of S, $T=\{t \in T'; h^1(\mathcal{S}'_t, -K_{\mathcal{S}'_t}) = b_2+1\}$,*

$\mathcal{S} = \tilde{\omega}^{-1}(T)$, $\pi = \tilde{\omega}_{\mathcal{S}}$: $\mathcal{S} \to T$ *where* T *is assumed to be reduced. Then there exist two divisors* $\mathcal{A}$ *and* $\mathcal{B}$ *of* $\mathcal{S}$ *flat over* T *such that*

(5.7.1) $(\mathcal{S}, \mathcal{A}, \mathcal{B})_0 \cong (S, A, B)$,

(5.7.2) $(\mathcal{S}, \mathcal{A}, \mathcal{B})$ *is versal at* 0 *in the sense that for any given smooth proper* f: $\mathcal{S}'' \to T''$ *with two divisors* $\mathcal{A}''$ *and* $\mathcal{B}''$ *flat over* T'' *such that* $(\mathcal{S}'', \mathcal{A}'', \mathcal{B}'')_0 \cong (S, A, B)$, *there exists a Cartesian diagram (by shrinking* T''*)*

$$\begin{tikzcd}
\mathcal{S}'' \arrow[r, "H"'] \arrow[d, "f"'] & \mathcal{S} \arrow[d, "\pi"] \\
T'' \arrow[r, "h"'] & T
\end{tikzcd}$$

such that $h(0) = 0, \mathcal{A}'' = H^*\mathcal{A}$, $\mathcal{B}'' = H^*\mathcal{B}$.

(5.7.3) *Let* t_0 *be any point of* T *sufficiently near to* 0. *Then* $(\mathcal{S}, \mathcal{A}, \mathcal{B})$ *is versal at* t_0 *in the sense of* (5.7.2). *In other words, for any given smooth proper* f: $\mathcal{S}'' \to T''$ *with two divisors* $\mathcal{A}''$ *and* $\mathcal{B}''$ *of* $\mathcal{S}''$ *flat over* T'' *such that* $(\mathcal{S}'', \mathcal{A}'', \mathcal{B}'')_0 \cong (\mathcal{S}, \mathcal{A}, \mathcal{B})_{t_0}$, *there exists a Cartesian diagram*

$$\begin{tikzcd}
\mathcal{S}'' \arrow[r, "H"'] \arrow[d, "f"'] & \mathcal{S} \arrow[d, "\pi"] \\
T'' \arrow[r, "h"'] & T
\end{tikzcd}$$

such that $h(0) = t_0, \mathcal{A}'' = H^*\mathcal{A}$, $\mathcal{B}'' = H^*\mathcal{B}$.

Proof. In what follows we shrink T' around 0 if necessary. Since $h^2(\mathcal{S}_t', -K_{\mathcal{S}_t'}) = h^0(\mathcal{S}_t', 2K_{\mathcal{S}_t'}) = 0$ for any $t \in T'$, we have $R^1\tilde{\omega}_*(-K_{\mathcal{S}'})_t \cong H^1(\mathcal{S}_t', -K_{\mathcal{S}_t'})$, and $h^1(\mathcal{S}_t', -K_{\mathcal{S}_t'}) = h^0(\mathcal{S}_t', -K_{\mathcal{S}_t'}) + b_2 = b_2$ or $b_2 + 1$ by $h^1(S, -K_S) = b_2 + 1$, $h^0(S, -K_S) = 1$ and the upper-semicontinuity. Hence T is an analytic subset of T' containing 0 and t is contained in T iff $h^0(\mathcal{S}_t, -K_{\mathcal{S}_t}) = 1$. The coherent sheaves $R^1\pi_*(-K_{\mathcal{S}})$ and $R^0\pi_*(-K_{\mathcal{S}})$ are free $\mathcal{O}_T$ modules of rank $b_2 + 1$ and one respectively. Then a generating section of $R^0\pi_*(-K_{\mathcal{S}})$ gives a divisor $\mathcal{D}$ of $\mathcal{S}$ flat over T such that $\mathcal{D}_0 = A + B$, $\mathcal{D}$ has two connected components $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{D} = \mathcal{A} + \mathcal{B}$. Let f: $\mathcal{S}'' \to T''$ be a smooth proper family with two T'' flat divisors $\mathcal{A}''$ and $\mathcal{B}''$ such that $(\mathcal{S}'', \mathcal{A}'', \mathcal{B}'')_0 \cong (S, A, B)$. We shall show $K_{\mathcal{S}''} + \mathcal{A}'' + \mathcal{B}'' = 0$ by shrinking T'' around 0 if necessary. Let $F = K_{\mathcal{S}''} + \mathcal{A}'' + \mathcal{B}''$. Then $F_t := F_{\mathcal{S}_t''}$ is a flat line bundle for any t ($t \in T''$) because F_0 is trivial. So F_t can be viewed as a $\mathbf{C}^*$-valued continuous function of t since $H^1(\mathcal{S}_t'', \mathbf{C}^*) \cong H^1(S, \mathbf{C}^*) \cong \mathbf{C}^*$. Moreover since $\mathcal{A}_t''$ is either an elliptic curve or a cycle of rational curves, $\mathrm{Ker}(H^1(\mathcal{S}_t'', \mathbf{C}^*) \to H^1(\mathcal{A}_t'', \mathcal{O}^*_{\mathcal{A}_t''}))$ is discrete. Since $F_{t|\mathcal{A}_t''} = (K_{\mathcal{S}_t''} + \mathcal{A}_t'')_{\mathcal{A}_t''} = \mathcal{O}_{\mathcal{A}_t''}$, $F_0 = 1$ (trivial), the function F_t reduces to a constant, hence identically trivial. Hence $F = f^*L$ for a line bundle L of T''. So $K_{\mathcal{S}''} + \mathcal{A}'' + \mathcal{B}'' = 0$ by taking T'' smaller, therefore $h^0(\mathcal{S}_t'', -K_{\mathcal{S}_t''}) = 1$ for any $t \in T''$. By the semi-universality of $\tilde{\omega}$, we have a Cartesian diagram

$$\begin{tikzcd}
\mathscr{S}'' \arrow[r, "H"'] \arrow[d, "f"'] & \mathscr{S}' \arrow[d, "\tilde{\omega}"] \\
T'' \arrow[r, "h"'] & T'
\end{tikzcd} .$$

Since $h^0(\mathscr{S}_t'', -K_{\mathscr{S}_t''}) = 1$, $h^1(\mathscr{S}_t'', -K_{\mathscr{S}_t''}) = b_2 + 1$, we have $h(T'') \subset T$, $H(\mathscr{S}'') \subset \mathscr{S}$. Then $H^*\mathscr{A} = \mathscr{A}''$, $H^*\mathscr{B} = \mathscr{B}''$ because $\mathcal{O}_{\mathscr{S}''}(-\mathscr{A}'' - \mathscr{B}'') \cong \mathcal{O}_{\mathscr{S}''}(K_{\mathscr{S}''}) \cong \Omega^2_{\mathscr{S}''/T''} \cong H^*\Omega^2_{\mathscr{S}/T} \cong H^*\mathcal{O}_{\mathscr{S}}(K_{\mathscr{S}}) \cong H^*\mathcal{O}_{\mathscr{S}}(-\mathscr{A} - \mathscr{B}) \cong \mathcal{O}_{\mathscr{S}''}(-H^*\mathscr{A} - H^*\mathscr{B})$ and $h^0(\mathscr{S}_t'', -K_{\mathscr{S}_t''}) = 1$ for any $t \in T''$. Next we shall prove (5.7.3). Let t_0 be a point of T sufficiently near to 0 so that $R^0\pi_*(-K_{\mathscr{S}}) \cong \mathcal{O}_T$, $R^1\pi_*(-K_{\mathscr{S}}) \cong \mathcal{O}_T^{\oplus b_2+1}$ near t_0. Let f: $\mathscr{S}'' \to T''$ be a proper smooth family with two T'' flat divisors $\mathscr{A}''$ and $\mathscr{B}''$ such that $(\mathscr{S}'', \mathscr{A}'', \mathscr{B}'')_0 \cong (\mathscr{S}, \mathscr{A}, \mathscr{B})_{t_0}$. By the same argument as in (5.7.2), we have $K_{\mathscr{S}''} + \mathscr{A}'' + \mathscr{B}'' = 0$. On the other hand $h^0(\mathscr{S}_t', \Theta_{\mathscr{S}_t'}) = h^2(\mathscr{S}_t', \Theta_{\mathscr{S}_t'}) = 0$, $h^1(\mathscr{S}_t', \Theta_{\mathscr{S}_t'}) = 2b_2$ is independent of t $(t \in T')$ so that $\tilde{\omega}$: $\mathscr{S}' \to T'$ is a semi-universal deformation of $\mathscr{S}_{t_0}$ if t_0 is sufficiently near to 0. Hence we have a Cartesian diagram

$$\begin{tikzcd}
\mathscr{S}'' \arrow[r, "H"'] \arrow[d, "f"'] & \mathscr{S}' \arrow[d, "\tilde{\omega}"] \\
T'' \arrow[r, "h"'] & T'
\end{tikzcd}$$

such that $h(0) = t_0$. By $h^1(\mathscr{S}_t'', -K_{\mathscr{S}_t''}) = h^0(\mathscr{S}_t'', -K_{\mathscr{S}_t''}) + b_2 = b_2 + 1$, we have $h(T'') \subset T$, $H(\mathscr{S}'') \subset \mathscr{S}$, and $H^*\mathscr{A} = \mathscr{A}''$, $H^*\mathscr{B} = \mathscr{B}''$ by the same argument as in (5.7.2). Q.E.D.

(5.8) **Lemma.** *Let* π: $\mathscr{X} \to T$ *be a smooth proper family in* (5.5), $U = (D')^2 \times (\mathbf{C}^*)^{a+b}$ $(a, b > 0)$, $\mathscr{Z} = \mathscr{X} \times (\mathbf{C}^*)^{a+b}$, p *the natural projection of* $\mathscr{Z}$ *onto* U, $\mathscr{E}_i$ *a divisor of* $\mathscr{Z}$ *defined by* $z_i = 0$ *(a smooth family of elliptic curves over* U*)* $(i = 1, 2)$. *Define sections of* $\mathscr{Z}$ *over* U $e_1, \cdots, e_a, f_1, \cdots, f_b$ *by*

$$e_i(t_1, t_2, s_1, \cdots, s_a, u_1, \cdots, u_b) = (t_1, t_2, 0, s_i, s_1, \cdots, s_a, u_1, \cdots, u_b),$$
$$f_j(t_1, t_2, s_1, \cdots, s_a, u_1, \cdots, u_b) = (t_1, t_2, u_j, 0, s_1, \cdots, s_a, u_1, \cdots, u_b).$$

Blow up $\mathscr{Z}$ *successively with centers* $e_1, \cdots, e_a, f_1, \cdots, f_b$ *to obtain a smooth proper family* $\tilde{\omega}$: $\mathscr{S} \to U$ *with two* U*-flat divisors* $\mathscr{A}$ *and* $\mathscr{B}$ *such that* $\mathscr{S}_u$ $(u \in U)$ *is a blown-up diagonal Hopf surface with two elliptic curves* $\mathscr{A}_u$ *and* $\mathscr{B}_u$ *and* $\mathscr{A}_u^2 = -a$, $\mathscr{B}_u^2 = -b$. *The family* $\tilde{\omega}$ *is versal at any* $u \in U$ *in the sense of* (5.7.3).

Proof. The argument in (5.7) can be applied to prove (5.8) if we check the following

(5.8.1) $K_{\mathscr{S}} = -\mathscr{A} - \mathscr{B}$,

(5.8.2) $h^0(\mathscr{S}_u, -K_{\mathscr{S}_u}) = 1$, $h^1(\mathscr{S}_u, -K_{\mathscr{S}_u}) = a + b + 1$ for any u,

(5.8.3) $h^q(\mathscr{S}_u, \Theta_{\mathscr{S}_u}) = 0$ $(q = 0, 2)$, $h^1(\mathscr{S}_u, \Theta_{\mathscr{S}_u}) = 2(a + b)$ for any u. (5.8.1) follows from $K_{\mathscr{Z}} = -\mathscr{E}_1 - \mathscr{E}_2$, and $e_i(U) \subset \mathscr{E}_1$ (or its proper transforms) and $f_j(U) \subset \mathscr{E}_2$ (or its proper transforms). By (5.8.1) $h^0(\mathscr{S}_u, -K_{\mathscr{S}_u}) \geqq 1$. Let $\mathscr{A}$ and $\mathscr{B}$ be proper transforms of $\mathscr{E}_1$ and $\mathscr{E}_2$ respectively. Since $\mathscr{A}_u$ and $\mathscr{B}_u$ are irreducible and $\mathscr{A}_u^2 < 0$, $\mathscr{B}_u^2 < 0$, we have $h^0(\mathscr{S}_u, -K_{\mathscr{S}_u}) = 1$. By Riemann-Roch and $h^2(\mathscr{S}_u, -K_{\mathscr{S}_u}) = 0$, we have $h^1(\mathscr{S}_u, -K_{\mathscr{S}_u}) = a+b+1$. Next we shall prove (5.8.3). First we have $h^2(\mathscr{S}_u, \Theta_{\mathscr{S}_u}) = h^0(\mathscr{S}_u, \Omega^1_{\mathscr{S}_u} \otimes \Omega^2_{\mathscr{S}_u}) = h^0(\mathscr{Z}_u, \Omega^1_{\mathscr{Z}_u} \otimes \Omega^2_{\mathscr{Z}_u}) = 0$. On the other hand $h^0(\mathscr{Z}_u, \Theta_{\mathscr{Z}_u}) = 2$ or 4 according as $t_1 \neq t_2$ or $t_1 = t_2$. It is easy to see that $\mathrm{Aut}^0(\mathscr{Z}_u) \cong (\mathbf{C}^*)^2$ or $GL(2, \mathbf{C})$. Hence $\mathrm{Aut}^0(\mathscr{S}_u) \cong \{1\}$ because $a, b > 0$. Therefore $h^0(\mathscr{S}_u, \Theta_{\mathscr{S}_u}) = 0$, and $h^1(\mathscr{S}_u, \Theta_{\mathscr{S}_u}) = 2(a+b)$ by Riemann-Roch. Q.E.D.

(5.9) *Proof of* (5.6). Let $\pi\colon \mathscr{S} \to T$ be the versal family with two T-flat divisors $\mathscr{A}$ and $\mathscr{B}$ in (5.7). Then $Q = \{t \in T; \mathscr{A}_t$ or $\mathscr{B}_t$ is singular$\}$ is an analytic subset of T of codimension one. On the other hand in the family $\omega\colon \mathscr{S} \to U$ in (5.8) $P := \{u \in U;\ \mathscr{S}_u$ has non-constant meromorphic functions$\}$ is $\{(t_1, t_2, s_1, \cdots, u_b);\ t_1^n = t_2^m$ for some positive integers n and $m\}$, a countable union of analytic subsets of U. Hence by taking a general curve in T passing through 0, we have a family $f\colon \mathscr{S}'' \to D$ with divisors $\mathscr{A}''$ and $\mathscr{B}''$ such that $(\mathscr{S}'', \mathscr{A}'', \mathscr{B}'')_0 \cong (S, A, B)$, $\mathscr{A}_t''$ and $\mathscr{B}_t''$ are nonsingular for $t \neq 0$ and $\mathscr{S}_u''$ has no meromorphic functions except constants for t contained in a dense subset D^* of D. In fact, $D^* = D$ with a countable subset deleted. Q.E.D.

(5.10) **Lemma.** *Let* $\pi\colon \mathscr{S} \to D$ *be the same as in* (5.4). *Suppose that there exists a dense subset* D^* *of* D *such that* $\mathscr{S}_t$ *has no meromorphic functions except constants for* $t \in D^*$. *Then there exist holomorphic functions* α_1 *and* α_2 *on* D, *sections* $e_1, \cdots, e_a$ *(resp.* $f_1, \cdots, f_b$*) of* $\mathscr{E}_1$ *(resp. of* $\mathscr{E}_2$*) or its proper transforms over* D' *where* $\mathscr{E}_i$ *is a divisor of* $\mathscr{X}(\alpha_1, \alpha_2)$ *defined by* $z_i = 0$ *such that*

(5.10.1) $0 < |\alpha_i(t)| < 1$ *for* $t \neq 0$, *order* α_1 *at* $t = 0 = \sharp$ *(irreducible components of* B*), order* α_2 *at* $t = 0 = \sharp$ *(irreducible components of* A*)*

(5.10.2) $\mathscr{S}' := \pi^{-1}(D')$ *is a blowing-up of* $\mathscr{X}(\alpha_1, \alpha_2)$ *with centers* $e_1, \cdots, e_a$ *and* $f_1, \cdots, f_b$. $\mathscr{A}' := \mathscr{S}' \cap \mathscr{A}$ *and* $\mathscr{B}' := \mathscr{S}' \cap \mathscr{B}$ *are proper transforms of* $\mathscr{E}_1$ *and* $\mathscr{E}_2$.

Proof. $\mathscr{S}_t (t \neq 0)$ is a blown-up diagonal Hopf surface by (5.4). Since exceptional curves of the first kind are stable by small deformations [13, Theorem 5], we have a smooth proper family $h\colon \mathscr{X} \to D'$ of minimal diagonal Hopf surfaces by blowing down exceptional curves simultaneously. Let $\mathscr{S}' = \pi^{-1}(D')$, $f\colon \mathscr{S}' \to \mathscr{X}$ be the natural mapping (blowing-down), $\bar{\mathscr{A}} = f(\mathscr{A} \cap \mathscr{S}')$, $\bar{\mathscr{B}} = f(\mathscr{B} \cap \mathscr{S}')$. Then $\bar{\mathscr{A}}_t$ and $\bar{\mathscr{B}}_t$ are nonsingular elliptic curves with $[\bar{\mathscr{A}}_t]$ and $[\bar{\mathscr{B}}_t]$ flat. Therefore $h^0(\mathscr{X}_t, \Omega^1_{\mathscr{X}_t}(\log \bar{\mathscr{A}}_t)) = h^0(\mathscr{X}_t, \Omega^1_{\mathscr{X}_t}(\log \bar{\mathscr{B}}_t)) = 1$ by the proof of (3.4), and by Riemann-Roch, $h^1(\mathscr{X}_t, \Omega^1_{\mathscr{X}_t}(\log \bar{\mathscr{A}}_t)) = h^1(\mathscr{X}_t, \Omega^1_{\mathscr{X}_t}(\log \bar{\mathscr{B}}_t)) = b_2 + 1$ even if $\mathscr{X}_t$ has nonconstant meromorphic functions. Hence $R^q h_* \Omega^1_{\mathscr{X}/D'}(\log \bar{\mathscr{A}})$ and $R^q h_* \Omega^1_{\mathscr{X}/D'}(\log \bar{\mathscr{B}})$ $(q = 0, 1)$ are free $\mathscr{O}_{D'}$ modules of rank one and $b_2 + 1$. Let s_1 and s_2 be generating sections of $R^0 h_* \Omega^1_{\mathscr{X}/D'}(\log \bar{\mathscr{A}})$ and $R^0 h_* \Omega^1_{\mathscr{X}/D'}(\log \bar{\mathscr{B}})$, γ a generator of $\pi_1(\mathscr{X}_t)$ ($\cong \mathbf{Z}$ by (5.4)) contained in $\mathscr{X}_t - \bar{\mathscr{A}}_t - \bar{\mathscr{B}}_t$. We define $\alpha_i(t) = \exp\left(2\pi i \int_\gamma s_i\right)$. Then $\alpha_i(t)$ is a holomorphic function on D'. By [12, II, p. 703] we have either $0 < |\alpha_i(t)| < 1$ $(i = 1, 2)$ or $|\alpha_i(t)| > 1$ $(i = 1, 2)$. We may assume $0 < |\alpha_i(t_0)| < 1$ for some t_0. Then it follows that $0 < |\alpha_i(t)| < 1$ for any t. Indeed, if there is t' with $|\alpha_i(t')| > 1$, then there is t'' on an arc connecting t_0 and t' such that $|\alpha_i(t'')| = 1$ which is a contradiction. Hence by (5.5) we have a smooth proper family $\bar{\pi}$: $\mathscr{X}(\alpha_1, \alpha_2) \to D'$. Next we want to define a holomorphic mapping F of $\mathscr{X}$ into $\mathscr{X}(\alpha_1, \alpha_2)$. For this we need the following claim.

(5.10.3) *$\mathscr{X} - \bar{\mathscr{A}} - \bar{\mathscr{B}}$ has a section o over D'.*

Proof. If U_j is a sufficiently small open subset of D', then we have a section o_j of $\mathscr{X} - \bar{\mathscr{A}} - \bar{\mathscr{B}}$ over U_j. Then we define a holomorphic mapping F_j of $\mathscr{X}$ into $\mathscr{X}(\alpha_1, \alpha_2)$ over U_j by $F_j(t,p) = \left(t, \exp\left(2\pi i \int_{o_j(t)}^{p} s_1\right), \exp\left(2\pi i \int_{o_j(t)}^{p} s_2\right)\right)$. By (5.4) $F_j(t,\ )$ is an isomorphism for t contained in $D^* \cap U_j$, hence for any t in U_j. Let G be a group generated by h_j, a transformation of $U_j \times (\mathbf{C}^2 - 0)$ defined by h_j: $(t, x_j, y_j) \to (t, \alpha_1(t)x_j, \alpha_2(t)y_j)$. $h^{-1}(U_j)$ is isomorphic to $U_j \times (\mathbf{C}^2 - 0)/G$. Therefore $\mathscr{X}$ is a union of $U_j \times (\mathbf{C}^2 - 0)/G$ where (t, x_j, y_j) and (t, x_k, y_k) are identical iff $x_j = f_{jk}(t)x_k$, $y_j = g_{jk}(t)y_k \bmod G$ for nonvanishing holomorphic functions f_{jk} and g_{jk} on $U_j \cap U_k$. (This is the case because D^* is dense in D.) Consider an exact sequence of sheaves of abelian groups, $0 \to \mathbf{Z} \xrightarrow{i} (\mathcal{O}_{D'}^*)^2 \to \mathscr{N} \to 1$ where i is defined by $i(n) = (\alpha_1(t)^n, \alpha_2(t)^n)$. Then a set of pairs $(f_{jk}(t), g_{jk}(t))$ defines a one cocycle on D' with coefficients in $\mathscr{N}$. Since $H^1(D', \mathscr{N}) \cong H^2(D', \mathbf{Z}) = 0$, there exist nonvanishing holomorphic functions f_j and g_j on U_j such that $(f_j, g_j) = (f_{jk}f_k, g_{jk}g_k) \bmod G$. Then $o = \bigcup_j \{(t, x_j, y_j) = (t, f_j(t), g_j(t))\}$ is a section of $\mathscr{X} - \bar{\mathscr{A}} - \bar{\mathscr{B}}$ over D' as desired. Thus the proof of (5.10.3) is complete.

Now we define a holomorphic mapping F of $\mathscr{X}$ into $\mathscr{X}(\alpha_1, \alpha_2)$ by $F(t,p) = \left(t, \exp\left(2\pi i \int_{o(t)}^{p} s_1\right), \exp\left(2\pi i \int_{o(t)}^{p} s_2\right)\right)$ where $o(t)$ is a section of $\mathscr{X} - \bar{\mathscr{A}} - \bar{\mathscr{B}}$ over D'. By (5.4) $F(t,\ )$ is an isomorphism if t is in D^*. Therefore $F(t,\ )$ is an isomorphism for any t, hence F is an isomorphism of $\mathscr{X}$ with $\mathscr{X}(\alpha_1, \alpha_2)$ over D'. Clearly $F(\bar{\mathscr{A}}) = \mathscr{E}_1$, $F(\bar{\mathscr{B}}) = \mathscr{E}_2$. By the definition of $\mathscr{X}$, we have sections $e_1, \cdots, e_a$ and $f_1, \cdots, f_b$ of $\mathscr{X}(\alpha_1, \alpha_2)$ such that $\mathscr{S}'$ is a succession of blowing-ups of $\mathscr{X}(\alpha_1, \alpha_2)$ with centers $e_1, \cdots, f_b$. By the proof of (5.7) we have $K_{\mathscr{S}} = -\mathscr{A} - \mathscr{B}$, so $e_i(D)$ and $f_j(D)$ are contained in $\bar{\mathscr{A}} + \bar{\mathscr{B}}$ or its proper transforms. By $\mathscr{A}_t^2 = -a$, $\mathscr{B}_t^2 = -b$, we may assume $e_i(D) \subset \bar{\mathscr{A}}$ or its proper transforms and $f_j(D) \subset \bar{\mathscr{B}}$ or its proper transforms. Both $\mathscr{A}$ and $\mathscr{B}$ are elliptic surfaces over D. Since $K_{\mathscr{A}} \cong (K_{\mathscr{S}} + \mathscr{A})_{\mathscr{A}} \cong \mathcal{O}_{\mathscr{A}}$, $K_{\mathscr{B}} \cong (K_{\mathscr{S}} + \mathscr{B})_{\mathscr{B}} \cong \mathcal{O}_{\mathscr{B}}$, both $\mathscr{A}$ and $\mathscr{B}$ are minimal along $\mathscr{A}_0$ and $\mathscr{B}_0$, hence by [11, II. p. 597], the order of α_1 at $t = 0$ equals $\sharp$ (irreducible components of B), the order of α_2 at $t = 0$ equals $\sharp$ (irreducible components of A). Q.E.D.

(5.11) **Lemma.** *Let S be a VII_0 surface with an elliptic curve E with $E^2 = -n$ and a cycle Z of rational curves. Then there exists a one-parameter smooth proper family π: $\mathscr{S} \to D$ with two divisors $\mathscr{E}$ and $\mathscr{Z}$ flat over D, holomorphic functions α_1, α_2 on D, n sections $e_1, \cdots, e_n$ of $\mathscr{E}_1$ (or proper transforms of $\mathscr{E}_1$) over D' where $\mathscr{E}_1$ is a divisor $z_1 = 0$ of $\mathscr{X}(\alpha_1, \alpha_2)$, $D' = D - 0$ such that*

(5.11.1) $(\mathscr{S}, \mathscr{E}, \mathscr{Z})_0 = (S, E, Z)$,

(5.11.2) $0 < |\alpha_1(t)| < 1$ $(t \neq 0)$, *order* α_1 *at* $t = 0 = \sharp$ *(irreducible components of* Z*),* $\alpha_2(t)$ *is constant and* $0 < |\alpha_2(t)| < 1$,

(5.11.3) *$\mathscr{S}_t$ has no meromorphic functions except constants for generic t,* (5.11.4) *$\mathscr{S}' := \pi^{-1}(D')$ is a blowing-up of $\mathscr{X}(\alpha_1, \alpha_2)$ along $e_i (1 \leqq i \leqq n)$, $\mathscr{E}$ and $\mathscr{Z}$ are proper transforms of $\mathscr{E}_1$ and $\mathscr{E}_2$ and $\mathscr{E}_t^2 = -n$, $\mathscr{Z}_t^2 = 0 \, (t \neq 0)$ where $\mathscr{E}_2$ is a divisor $z_2 = 0$ of $\mathscr{X}(\alpha_1, \alpha_2)$.*

Proof. By (4.6) there exists a one-parameter family $\pi\colon \mathscr{S} \to D$ with two divisors $\mathscr{E}$ and $\mathscr{Z}$ flat over D such that $(\mathscr{S}, \mathscr{E}, \mathscr{Z})_0 \cong (S, E, Z)$, and $\mathscr{E}_t \cong E$, $\mathscr{Z}_t$ is a smooth elliptic curve for $t \neq 0$. By the argument similar to (5.7) we may assume that $\mathscr{S}_t$ has no meromorphic functions except constants for generic t by choosing a general π. Then by the same argument as in (5.10) $\mathscr{S}' = \pi^{-1}(D')$ is a blowing-up of $\mathscr{X}(\alpha_1, \alpha_2)$ along n sections $e_1, \cdots, e_n$ of $\mathscr{E}_1$ (or proper transforms of $\mathscr{E}_1$) where α_i are bounded holomorphic functions on D. Since $\mathscr{E}_t^2 = E^2 = -n$, $\mathscr{Z}_t^2 = Z^2 = 0$, the centers of blowing-ups are on $\mathscr{E}_1$ or the proper transforms of $\mathscr{E}_1$. Since the periods of $\mathscr{E}_t$ are $2\pi i$ and $\log \alpha_2(t)$, $\alpha_2(t)$ must be constant, and we may assume $0 < |\alpha_2(t)| < 1$. Since $\mathscr{Z}$ is an elliptic surface over D and $K_{\mathscr{Z}} \cong (K_{\mathscr{S}} + \mathscr{Z})_{\mathscr{Z}} = \mathcal{O}_{\mathscr{Z}}$ as in (5.7), $\mathscr{Z}$ is minimal along $\mathscr{Z}_0 = Z$. Hence the bounded holomorphic function $\alpha_1(t)$ has a zero of order $\#$ (irreducible components of Z) at $t = 0$. Q.E.D.

§6. Duality

(6.1) **Theorem (Duality I).** *Let S be a* VII$_0$ *surface with two cycles A and B of rational curves. Then*

$$A^2 = -\#\ (\textit{irreducible components of } B),$$
$$B^2 = -\#\ (\textit{irreducible components of } A),$$

and $b_2 = \#$ (irreducible components of $A + B$), there are no curves other than irreducible components of A and B.

(6.2) Let $a = -A^2$, $b = -B^2$, $r = \#$ (irreducible components of A), $s = \#$ (irreducible components of B) and let $A = \sum_{\lambda=1}^{r} A_\lambda$, $B = \sum_{\nu=1}^{s} B_\nu$ be the decompositions of A and B into irreducible components such that

$$A_\lambda A_\nu = \begin{cases} 1 & (\lambda = \nu \pm 1 \bmod r), \\ 0 & (\lambda \neq \nu, \nu \pm 1 \bmod r), \end{cases} \qquad B_\lambda B_\nu = \begin{cases} 1 & (\lambda = \nu \pm 1 \bmod s) \\ 0 & (\lambda \neq \nu, \nu \pm 1 \bmod s). \end{cases}$$

We recall $K_S = -A - B$, $b_2 = -c_1^2 = -A^2 - B^2 = a + b$ by (2.1.2) and (2.8).

(6.3) **Lemma.** *Let N be a line bundle on S with $NA > 0$* or *$NB > 0$. Then $h^0(S, N) = 0$.*

Proof. Without loss of generality we may assume $NA > 0$. Assume $H^0(S, N) \neq 0$. Then there exists an effective divisor E such that $[E] = N$. By (2.9) we can express $E = E_1 + E_2 + E_3$ with $\operatorname{supp}(E_1) \subset A$, $\operatorname{supp}(E_2) \subset B$, $\operatorname{supp}(E_3) \cap (A \cup B) = \phi$. By assumption $E_1 A > 0$. Let $E_1 = \sum_{\lambda=1}^{r} n_\lambda A_\lambda$. Assume $r > 1$. Then $E_1 A = \sum_{\lambda=1}^{r} (2 + A_\lambda^2) n_\lambda \leqq 0$ which is absurd. So $r = 1$. Then $E_1 = nA$, $E_1 A = nA^2 \leqq 0$ by (2.12) which is again absurd. Q.E.D.

(6.4) **Lemma.** *Let L be a line bundle on S with $L^2 = K_S L = -1$, $LA = 1$. Suppose that $A^2 \neq -1$. Then there exists an irreducible component B_ν of B such that $LB_\nu \neq 0$.*

Proof. Assume that $LB_\nu = 0$ for any irreducible component B_ν of B. Then L_B (the restriction of L to B) is flat. Since $H^1(S, \mathbf{C}^*) \cong H^1(B, \mathbf{C}^*)$ by (2.14), there exists a flat line bundle F on S such that $F_B = L_B$. Hence by taking $L - F$ instead of L we may assume L_B to be trivial. By $LA = 1$ and (6.3) we have $h^0(S, L) = 0$. Hence $h^0(S, L - B) = 0$. By Riemann-Roch and Serre duality, we have $h^1(S, L - B) = h^2(S, L - B) + (-K_S L + K_S B + (L - B)^2)/2 = h^2(S, L - B) = h^0(S, -L - A)$. Since $A^2 \neq -1$, we have $(-L - A)A > 0$. Hence by (6.3) $h^0(S, -L - A) = 0$. Therefore $h^0(B, L_B) \leqq h^0(S, L) + h^1(S, L - B) = 0$ which contradicts $h^0(B, L_B) = h^0(B, \mathcal{O}_B) = 1$. Q.E.D.

(6.5) **Lemma.** *There exist line bundles on S, L_j, M_k $(1 \leqq j \leqq a, 1 \leqq k \leqq b)$ such that*

(6.5.1) $A = -\sum_{j=1}^{a} L_j$, $B = -\sum_{k=1}^{b} M_k$ in $H^2(S, \mathbf{Z})$

(6.5.2) $L_j^2 = M_k^2 = -1$, $K_S L_j = K_S M_k = -1$, $L_i L_j = L_j M_k = M_k M_l = 0$ $(i \neq j, k \neq l)$ *where* $a = -A^2$, $b = -B^2$.

Proof. By (5.4) there exists line bundles L'_j and M'_k on $\mathcal{S}_t (t \neq 0)$ $(1 \leqq j \leqq a, 1 \leqq k \leqq b)$ such that $\mathcal{A}_t = -\sum_{j=1}^{a} L'_j$, $\mathcal{B}_t = -\sum_{k=1}^{b} M'_k$ in $H^2(\mathcal{S}_t, \mathbf{Z})$ and $L_j'^2 = M_k'^2 = -1$, $K_{\mathcal{S}_t} L'_j = K_{\mathcal{S}_t} M'_k = -1$, $L'_i L'_j = L'_j M'_k = M'_k M'_l = 0$ $(i \neq j, k \neq l)$. Since $H^2(\mathcal{S}_t, \mathbf{Z}) \cong H^2(S, \mathbf{Z})$, $H^1(S, \mathcal{O}_S^*) \to H^2(S, \mathbf{Z})$ is surjective, there are line bundles L_j and M_k on S as desired.

(6.6) *Proof of* (6.1). Suppose first $\min(a, b, r, s) \geqq 2$. Since L_j and M_k is a $\mathbf{Z}$ basis of $H^2(S, \mathbf{Z})$, we can express

$$A_\lambda = \sum_{j=1}^{a} n_{\lambda,j} L_j + \sum_{k=1}^{b} m_{\lambda,k} M_k,$$

$$B_\lambda = \sum_{j=1}^{a} n'_{\lambda,j} L_j + \sum_{k=1}^{b} m'_{\lambda,k} M_k$$

for certain integers $n_{\lambda,j}, n'_{\lambda,j}, m_{\lambda,k}, m'_{\lambda,k}$. By applying (6.4) to $L = L_j$, there exists at least one $n'_{\lambda,j} \neq 0$ for any j. By (6.5.2) $B = -\sum_{k=1}^{b} M_k$, there are at least two $n'_{\lambda,j} \neq 0$ for any j. On the other hand there is at least one $m'_{\lambda,k} \neq 0$ for any k. Therefore

$$\begin{aligned} b = -B^2 &= -\sum_{\lambda=1}^{s} B_\lambda^2 - 2s \\ &= \sum_{\lambda,j} (n'_{\lambda,j})^2 + \sum_{\lambda,k} (m'_{\lambda,k})^2 - 2s \geqq 2a + b - 2s. \end{aligned}$$

Hence $s \geqq a$. Similarly $r \geqq b$. However since the intersection matrices $(A_\lambda A_\nu)$ and $(B_\lambda B_\nu)$ are negative definite in view of (2.12), A_λ and B_ν $(1 \leqq \lambda \leqq r, 1 \leqq \nu \leqq s)$ are linearly independent in $H^2(S, \mathbf{Z})$. In other words $b_2 \geqq r + s$. Since $b_2 = a + b$, this implies that $b_2 = r + s$, $r = b$, $s = a$.

If $\min(a, b, r, s) = 1$, then by taking a double covering $\pi\colon S^* \to S$, we have a VII_0 surface S^* with two cycles A^* and B^*, $A^* = \pi^{-1}(A)$, $B^* = \pi^{-1}(B)$. Then $(A^*)^2 = 2A^2$, $(B^*)^2 = 2B^2$, $\sharp$ (irreducible components of A^*) $= 2\sharp$ (irreducible components of A), $\sharp$ (irreducible components of B^*) $= 2\sharp$ (irreducible components of B). So we can apply the above result to S^* and have $r = b$, $s = a$. The remaining assertion of (6.1) is now clear from (2.9). Q.E.D.

(6.7) **Corollary.** $m_{\lambda,k}$, $n'_{\lambda,j} = \pm 1$, or 0, $n_{\lambda,j}$, $m'_{\lambda,k} = -1$, or 0

$$\sharp\{n_{\lambda,j} \neq 0;\ 1 \leqq \lambda \leqq r\} = 1,\quad \sharp\{m_{\lambda,k} \neq 0;\ 1 \leqq \lambda \leqq r\} = 2$$
$$\sharp\{n'_{\lambda,j} \neq 0;\ 1 \leqq \lambda \leqq s\} = 2,\quad \sharp\{m'_{\lambda,k} \neq 0;\ 1 \leqq \lambda \leqq s\} = 1$$

for any j and k.

Clear.

(6.8) **Theorem (Duality II).** *Let S be a VII_0 surface with two cycles A and B of rational curves. Then there exist positive integers p_j, $q_j (\geqq 3)$ such that*

$$Zykel(A) = (p_1, \underbrace{2, \cdots, 2}_{(q_1-3)}, p_2, \underbrace{2, \cdots, 2}_{(q_2-3)}, \cdots\cdots, p_n, \underbrace{2, \cdots, 2}_{(q_n-3)})$$

$$Zykel(B) = (\underbrace{2, \cdots, 2}_{(p_1-3)}, q_1, \underbrace{2, \cdots, 2}_{(p_2-3)}, q_2, \cdots\cdots, q_{n-1}, \underbrace{2, \cdots, 2}_{(p_n-3)}, q_n).$$

More precisely, by renumbering L_j and M_k if necessary,

$$A_\lambda = M_{\lambda-1} - M_\lambda - \sum_{j \in I_\lambda} L_j,\quad B_\nu = L_\nu - L_{\nu+1} - \sum_{k \in J_\nu} M_k$$

where

$$I_\lambda = \begin{cases} \{m \in \mathbf{Z};\ R_j + 1 \leqq m \leqq R_{j+1}\} & (\lambda = S_j + 1 \textit{ for some } j (0 \leqq j \leqq n-1)) \\ \phi & (\textit{otherwise}) \end{cases}$$

$$J_\nu = \begin{cases} \{m \in \mathbf{Z};\ S_{j-1} + 1 \leqq \mathrm{m} \leqq S_j\} & (\nu = R_j \textit{ for some } j (1 \leqq j \leqq n)) \\ \phi & (\textit{otherwise}) \end{cases}$$

$(1 \leqq \lambda \leqq r, 1 \leqq \nu \leqq s)$ *and* $R_j = \sum_{i=1}^{j} (p_i - 2)$, $S_j = \sum_{i=1}^{j} (q_i - 2)$, $R_n = s$. (See (1.4) for the definition of $Zykel(\ )$.)

Proof. In the following proof we denote $\{m \in \mathbf{Z};\ a \leqq m \leqq b\}$ by $[a, b]$. By (6.7) we have $A_\lambda A_\nu = -\sum_{k=1}^{r} m_{\lambda,k} m_{\nu,k}$, $B_\lambda B_\nu = -\sum_{j=1}^{s} n'_{\lambda,j} n'_{\nu,j}$. (Notice that $r = b$, $s = a$.) Since $A_\lambda A_{\lambda+1} = B_\nu B_{\nu+1} = 1$, there are at least one $m_{\lambda,k} (\neq 0)$ and at least one $n'_{\nu,j} (\neq 0)$ for any λ and ν. Since $A_\lambda B = B_\nu A = 0$, we have $\sum_{k=1}^{r} m_{\lambda,k} = 0$, $\sum_{j=1}^{s} n'_{\nu,j} = 0$ for any λ and ν therefore $\sum_{k=1}^{r} {m_{\lambda,k}}^2 \geqq 2$, $\sum_{j=1}^{s} {n'_{\nu,j}}^2 \geqq 2$. On the other hand by (6.7) or the proof of (6.6) we have $\sum_{\lambda,k} {m_{\lambda,k}}^2 = 2r$, $\sum_{\nu,j} {n'_{\nu,j}}^2 = 2s$. Hence $\sum_{k=1}^{r} {m_{\lambda,k}}^2 = 2$, $\sum_{j=1}^{s} {n'_{\nu,j}}^2 = 2$. This implies that there are exactly two nonzero $m_{\lambda,k}$ and two nonzero

$n'_{\nu,j}$ with distict sings for any λ and ν. Therefore by reordering L_j and M_k suitably, and by the relations $A_\lambda A_{\lambda+1} = B_\alpha B_{\alpha+1} = 1$, $A_\lambda A_\nu = B_\alpha B_\beta = 0$ ($\lambda \neq \nu$, $\nu \pm 1 \bmod r$; $\alpha \neq \beta$, $\beta \pm 1 \bmod s$), we can express

$$A_\lambda = M_{\lambda-1} - M_\lambda - \sum_{j\in I_\lambda} L_j$$

$$B_\nu = L_\nu - L_{\nu+1} - \sum_{k\in J_\nu} M_k$$

where $I_1 \amalg \cdots \amalg I_r$ (disjoint union) $= [1,s]$, $J_1 \amalg \cdots \amalg J_s$ (disjoint union) $= [1,r]$. We may assume, by a cyclic permutation of irreducible components of A,

$$Zykel(A) = (p_1, \underbrace{2,\cdots,2}_{(q_1-3)}, p_2, \underbrace{2,\cdots,2}_{(q_2-3)}, \cdots\cdots, p_n, \underbrace{2,\cdots,2}_{(q_n-3)}).$$

This is equivalent to the following: $\sharp(L_{S_j+1}) = R_{j+1} - R_j = p_{j+1} - 2$, $I_\lambda = \phi$ if $\lambda \neq S_j + 1$ for any j where $S_j = (q_1 - 2) + \cdots + (q_j - 2)$. Let N be the set of all ν with $J_\nu \neq \phi$, $N = \{T_1, T_2, \cdots, T_m\}$ where $T_1 < T_2 < \cdots < T_m$. Set $I^j = I_{S_j+1}$, $J^k = J_{T_k}$. For any ν with $T_k + 1 \leqq \nu \leqq T_{k+1} - 1$, we have $B_\nu = L_\nu - L_{\nu+1}$, hence by $A_{S_j+1}B_\nu = 0$, I^j contains ν iff I^j contains $\nu+1$. This implies that I^j contains $[T_k+1, T_{k+1}]$ or $I^j \cap [T_k+1, T_{k+1}] = \phi$. Since I^j is nonempty and the disjoint union of $[T_k+1, T_{k+1}]$ is $[1,s]$, I^j contains $[T_{k(j)}+1, T_{k(j)+1}]$ for some $k(j)$ and $m \geqq n$. By the same argument, J^k contains $[S_{j(k)}+1, S_{j(k)+1}]$ for some $j(k)$ and $n \geqq m$. Hence we have $m = n$ and $I^j = [T_{k(j)}+1, T_{k(j)+1}]$, $J^k = [S_{j(k)}+1, S_{j(k)+1}]$. We may suppose, without loss of generality, that $I^0 = [1, R_1]$. We shall show that $k(j) = j$, $j(k) = k-1$, $T_j = R_j$. By $A_{S_j+1}B_{T_k} = 0$,

$$\sharp(\{S_j\} \cap J^k) - \sharp(\{S_j+1\} \cap J^k) + \sharp(I^j \cap \{T_k\}) - \sharp(I^j \cap \{T_k+1\}) = 0.$$

This is equivalent to the following

(6.8.1) $j = j(k)$ iff $k = k(j) + 1$,

(6.8.2) $j = j(k) + 1$ iff $k = k(j)$.

By $A_1 = M_0 - M_1 - (L_1 + \cdots + L_{R_1})$, $A_1 B_\lambda = 0$, we have that $B_\lambda = L_\lambda - L_{\lambda+1}$ $(1 \leqq \lambda \leqq R_1 - 1)$, J_{R_1} contains 1. Hence $T_1 = R_1$, $J^1 = [1, S_1]$, $j(1) = 0$. By (6.8.2) $1 = j(1) + 1$ implies that $1 = k(1)$. We shall show by induction on l that $k(l) = l$, $j(l) = l - 1$. We assume $j(l) = l - 1$. Then by (6.8.2) $l = k(l)$. By (6.8.1) $l + 1 = k(l) + 1$ implies that $l = j(l+1)$. Consequently $I^j = [T_j + 1, T_{j+1}]$, $J^k = [S_{k-1}+1, S_k]$. Since $\sharp(I^j) = T_{j+1} - T_j = R_{j+1} - R_j$ and $T_1 = R_1$, we have $T_j = R_j$ for any j. Thus $B_{R_j}^2 = -2 - (S_j - S_{j-1}) = -q_j$,

$$Zykel(B) = (\underbrace{2,\cdots,2}_{(p_1-3)}, q_1, \cdots, q_{n-1}, \underbrace{2,\cdots,2}_{(p_n-3)}, q_n).$$

Q.E.D.

(6.9) **Theorem (Duality III).** *Let S be a* VII$_0$ *surface with two cycles A and B of rational curves. Then $H_2(A,\mathbf{Z})$ and $H_2(B,\mathbf{Z})$ are primitive sublattices of the unimodular lattice $H_2(S,\mathbf{Z})$, each being the orthogonal complement of the other. And we have,* $|\det(A_\lambda A_\nu)| = |\det(B_\lambda B_\nu)|$.

Proof. We have an exact sequence,

$$\begin{aligned}
0 &\to H_4(S-A,\mathbf{Z}) \to H_4(S,\mathbf{Z}) \xrightarrow{f} H_4(S,S-A,\mathbf{Z})\\
&\to H_3(S-A,\mathbf{Z}) \to H_3(S,\mathbf{Z}) \xrightarrow{g} H_3(S,S-A,\mathbf{Z})\\
&\to H_2(S-A,\mathbf{Z}) \to H_2(S,\mathbf{Z}) \xrightarrow{h} H_2(S,S-A,\mathbf{Z})\\
&\to H_1(S-A,\mathbf{Z}) \to H_1(S,\mathbf{Z}) \longrightarrow H_1(S,S-A,\mathbf{Z}) \to 0\,.
\end{aligned}$$

We notice that $H_k(S,S-A,\mathbf{Z}) \cong H^{4-k}(A,\mathbf{Z})$ by the generalized Poincarè duality. Hence f is an isomorphism, and $H_1(S,S-A,\mathbf{Z}) = 0$. And g is an isomorphism too. In fact, since $H_3(S,\mathbf{Z}) \cong H^1(S,\mathbf{Z})$ and g is equivalent to the natural homomorphism $H^1(S,\mathbf{Z}) \to H^1(A,\mathbf{Z})$ which is an isomorphism by (2.14). Now we shall show h is an epimorphism. By (6.5) $H_2(S,\mathbf{Z})$ is generated by (Poincarè duals of) L_j and M_k $(1 \leqq j \leqq s,\ 1 \leqq k \leqq r)$ and by (6.8) we have

$$\begin{aligned}
h(M_k) &= \sum_{\lambda=1}^{r} (M_k A_\lambda)\,[A_\lambda]^* = [A_k]^* - [A_{k+1}]^*\\
h(L_j) &= [A_{S_k+1}]^* \quad (R_k + 1 \leqq j \leqq R_{k+1})\,.
\end{aligned}$$

Consequently $\mathrm{Im}(h)$ contains all $[A_\lambda]^*$, hence h is an epimorphism. Therefore $H_1(S-A,\mathbf{Z}) \cong H_1(S,\mathbf{Z})$ and $H_2(S-A,\mathbf{Z})$ is the orthogonal complement $H_2(A,\mathbf{Z})^\perp$ of $H_3(A,\mathbf{Z})$ in $H_2(S,\mathbf{Z})$ with respect to the intersection product. Clearly $H_2(B,\mathbf{Z})$ is a subspace of $H_2(S,\mathbf{Z})$ by the natural homomorphism, orthogonal to $H_2(A,\mathbf{Z})$. Hence $H_2(B,\mathbf{Z})$ is contained in $H_2(A,\mathbf{Z})^\perp$. We shall show $H_2(B,\mathbf{Z}) = H_2(A,\mathbf{Z})^\perp$. Since $\mathrm{rank}\, H_2(B,\mathbf{Z}) = \mathrm{rank}\, H_2(A,\mathbf{Z})^\perp$, any element P of $H_2(A,\mathbf{Z})^\perp$ can be written as $P = \sum_{\lambda=1}^{s} a_\lambda B_\lambda$ for certain rational numbers a_λ. By the same argument as above we can show the surjectivity of the homomorphism of $H_2(S,\mathbf{Z})$ into $H_2(S,S-B,\mathbf{Z}) \cong H^2(B,\mathbf{Z})$. So there exists N_λ in $H_2(S,\mathbf{Z})$ such that $N_\lambda B_\nu = \delta_{\lambda\nu}$. Hence $a_\lambda = PN_\lambda$ is an integer and therefore P is in $H_2(B,\mathbf{Z})$. Thus $H_2(S-A,\mathbf{Z}) = H_2(A,\mathbf{Z})^\perp = H_2(B,\mathbf{Z})$. Moreover since h is surjective, we have $H_2(S,\mathbf{Z})/H_2(B,\mathbf{Z}) = H_2(S,\mathbf{Z})/H_2(S-A,\mathbf{Z}) \cong H_2(S,S-A,\mathbf{Z}) \cong H^2(A,\mathbf{Z})$, therefore $H_2(B,\mathbf{Z})$ is a primitive sublattice (that is, $H_2(S,\mathbf{Z})/H_2(B,\mathbf{Z})$ is free). Similarly $H_2(A,\mathbf{Z})$ is also a primitive sublattice of $H_2(S,\mathbf{Z})$. On the other hand since $H^2(A,\mathbf{Z})$ and $H_2(B,\mathbf{Z})$ are dual lattices of $H_2(A,\mathbf{Z})$ and $H_2(B,\mathbf{Z})$ respectively, we have

$$\begin{aligned}
|\det(A_\lambda A_\nu)| &= [H^2(A,\mathbf{Z}) : H_2(A,\mathbf{Z})]\\
&= [H^2(B,\mathbf{Z}) : H_2(B,Z)] = |\det(B_\lambda B_\nu)|\,. \qquad \text{Q.E.D.}
\end{aligned}$$

(6.10) **Corollary.** $H_1(S-A,\mathbf{Z}) \cong H_1(S-B,\mathbf{Z}) \cong H_1(S,\mathbf{Z}) \cong \mathbf{Z}\,.$

Proof. Since $H_1(S,S-A,\mathbf{Z}) \cong H^3(A,\mathbf{Z}) = 0$ and h is surjective, we have $H_1(S-A,\mathbf{Z}) \cong H_1(S,\mathbf{Z})$. By (2.14), $H_1(S,\mathbf{Z}) \cong H_1(A,\mathbf{Z}) \cong \mathbf{Z}$. The assertions for B follow in the same manner. Q.E.D.

(6.11) **Lemma.** *Let S be a VII_0 surface with an elliptic curve E and a cycle Z or rational curves. Then $E^2 = -\#$ (irreducible components of Z) $= -b_2$, and there are no curves other than E and irreducible components of Z.*

For the proof of (6.11) we need three lemmas that can be verified in the same manner as (6.3) (6.4) and (6.5).

(6.12) **Lemma.** *Let M be a line bundle on S with $ME > 0$. Then $h^0(S, M) = 0$.*

(6.13) **Lemma.** *Let M be a line bundle on S such that $M^2 = KM = -1$, $ME = 1$. Then $E^2 = -1$ or there exists an irreducible component Z_λ of Z such that $Z_\lambda M \neq 0$.*

(6.14) **Lemma.** *There exist line bundles M_k $(1 \leqq k \leqq b_2)$ on S such that*

(6.14.1) $$H^2(S, \mathbf{Z}) = \bigoplus_{k=1}^{b_2} \mathbf{Z}\, c(M_k),$$

(6.14.2) $$E = -(M_1 + M_2 + \cdots + M_{b_2}),$$

(6.14.3) $$M_k^2 = K_S M_k = -1,\ M_k M_l = 0 \quad (k \neq l).$$

(6.15) *Proof of* (6.11). Let n be $\sharp$(irreducible component of Z), $Z = \sum_{\lambda=1}^{n} Z_\lambda$ the decomposition of Z into irreducible components. Assume $E^2 \neq -1$, $n \neq 1$. By (6.14) we can express $Z_\lambda = \sum_{k=1}^{b_2} m_{\lambda,k} M_k$ with integers $m_{\lambda,k}$. By (6.13) there is at least one $m_{\lambda,k}(\neq 0)$ for any k. Since $\sum_{\lambda=1}^{n} Z_\lambda = Z = 0$ in $H^2(S, \mathbf{Z})$, we have at least two $m_{\lambda,k}(\neq 0)$ for any k. Hence $\sum_{\lambda,k} m_{\lambda,k}^2 \geqq \sharp\{m_{\lambda,k} \neq 0;\ 1 \leqq \lambda \leqq n,\ 1 \leqq k \leqq b_2\} \geqq 2b_2$. On the other hand $0 = Z^2 = -\sum_{\lambda,k} m_{\lambda,k}^2 + 2n$. From this we infer $n \geqq b_2$. However rank $(Z_\lambda Z_\nu)$ is equal to $n-1$, and $EZ_\lambda = 0$, $E^2 = -b_2 < 0$, which shows $b_2 \geqq n$, so $b_2 = n$, $m_{\lambda,k} = 0, \pm 1$. If $E^2 = -1$ or $n = 1$, then by taking a double covering $\pi: S^* \to S$ of S, we have $b_2(S^*) = \sharp$ (irreducible components of $\pi^{-1}(Z)) = 2n$, $b_2(S^*) = 2b_2(S)$, so $b_2 = n$. Q.E.D.

(6.16) **Corollary.** *With the notations in* (6.15), *by reordering M_k suitably, $E = -\sum_{k=1}^{n} M_k$, $Z_\lambda = M_\lambda - M_{\lambda+1}$ $(n > 1)$ or $Z_1 = 0$ $(n = 1)$.*

Proof. We may assume $n \neq 1$. By (6.15) $m_{\lambda,k} = 0$ or ± 1. By $Z_\lambda^2 = -2$, $\sharp\{m_{\lambda,k} \neq 0;\ 1 \leqq k \leqq n\} = 2$. By $Z_\lambda E = 0$, we have λ and λ' for any k such that $m_{\lambda,k} = 1$, $m_{\lambda',k} = -1$. By $Z_\lambda Z_{\lambda+1} = 1$. We have our assertion by reordering M_k if necessary. Q.E.D.

(6.17) **Corollary.** $H_1(S - E, \mathbf{Z}) \cong H_1(S, \mathbf{Z}) \cong \mathbf{Z}$, $H_1(S - Z, \mathbf{Z}) \cong \mathbf{Z}^2$.

Proof. There is an exact sequence;

$$\begin{aligned} &\to H_2(S - E, \mathbf{Z}) \to H_2(S, \mathbf{Z}) \xrightarrow{h} H_2(S, S - E, \mathbf{Z}) \\ &\to H_1(S - E, \mathbf{Z}) \to H_1(S, \mathbf{Z}) \longrightarrow H_1(S, S - E, \mathbf{Z}) \to 0\,. \end{aligned}$$

By the isomorphism $H_k(S, S - E, \mathbf{Z}) \cong H^{4-k}(E, \mathbf{Z})$, h is given by $h(M_k) = -[E]^* \in H^2(E, \mathbf{Z}) = \mathbf{Z}[E]^*$ for any k. Hence $H_1(S - E, \mathbf{Z}) \cong H_1(S, \mathbf{Z}) \cong \mathbf{Z}$ by (2.14). Similarly $H_2(S - Z, \mathbf{Z}) = \mathbf{Z}[E]$, $H_2(S, Z) = \bigoplus_{k=1}^{n} \mathbf{Z}[M_k]$, $H_2(S, S - Z, \mathbf{Z}) \cong H^2(Z, \mathbf{Z}) = \bigoplus_{\lambda=1}^{n} \mathbf{Z}[Z_\lambda]^*$, $H_1(S - Z, \mathbf{Z}) \cong H_1(S, \mathbf{Z}) \oplus \mathbf{Z}[Z_1]^*$. Q.E.D.

§7. Parabolic Inoue surfaces

The purpose of this section is to prove the following

(7.1) **Theorem.** *Let S be a* VII_0 *surface with an elliptic curve and a cycle of rational curves. Then S is isomorphic to a parabolic Inoue surface $S(t, n)$, $n = b_2(S)$.*

(7.2) Let S be a VII_0 surface with an elliptic curve E with $E^2 = -n$, and a cycle Z of rational curves. Then S has no meromorphic functions except constants, and $K_S + E + Z = 0$, $b_2 = n$, $Z^2 = 0$, $EZ = 0$ by (2.8), (2.12), (6.11). By (2.5) the line bundle $[Z]$ is flat. Hence we can choose an open covering U_j of S, equations $w_j = 0$ defining $Z \cap U_j$ such that $w_j = f_{jk} w_k$, $f_{jk} \in \mathbf{C}^*$, and the one cocycle f_{jk} represents $[Z]$. Then we define a meromorphic 1 form φ on S by $\varphi = w_j^{-1} dw_j$ (on U_j). Since $EZ = 0$, φ is holomorphic on a neighborhood on E.

(7.3) **Lemma.** *The restriction φ_E of φ to E is a nontrivial holomorphic 1 form on E.*

Proof. By the same argument as in (2.14), we can show that $i_*\colon H_1(E, \mathbf{Z}) \to H_1(S, \mathbf{Z})$ is surjective. If the restriction φ_E is zero, then we have for any $\gamma \in H_1(E, \mathbf{Z})$

$$\int_{i_*(\gamma)} \varphi = \int_{\gamma} \varphi_E = 0\,.$$

Therefore $\exp\left(\int_o^p \varphi\right)$ is a nontrivial holomorphic function of p ($p \in S$) where o is a point of $S - Z$. It is a contradiction. Q.E.D.

(7.4) By (5.11) $\pi_1(S) \cong \mathbf{Z}$. Let $\pi\colon \tilde{S} \to S$ be the universal covering of S. Let γ be a generator of $\pi_1(S)$ which corresponds naturally to a transformation g of $\tilde{S}$. Let $W = W(p) = \exp\left(\int_o^p \varphi\right)$. Then W is a single-valued holomorphic function on $\tilde{S}$ and satisfies $g^* W = \beta W$ for a constant $\beta\,(\neq 0)$. If $|\beta| = 1$, then $|W|$ is a C^∞ function on S satisfying the maximum principle. Since S is compact, this implies that W is constant which is a contradiction. Therefore $|\beta| \neq 1$. We may assume $|\beta| < 1$ by taking g^{-1} instead of g if necessary. Since $K + E + Z = 0$, there exists a meromorphic 2-form ψ with poles along $E + Z$, i.e. $(\psi) = -E - Z$. We may assume $\mathrm{Res}_E \psi$ (= the residue of ψ along E) $= \varphi_E$. Moreover via the natural isomorphism $\Omega_S^1(Z) \cong \Theta_S(K_S + Z) \cong \Theta_S(-E)$ induced by the nondegenerate pairings $\Omega_S^1(E) \times \Omega_S^1(Z) \to \Omega_S^2(E + Z) = \mathcal{O}_S \psi \cong \mathcal{O}_S$, and $\Omega_S^1(E) \times \Theta_S(-E) \to \mathcal{O}_S$, we have a holomorphic vector field $\theta\,(\neq 0)$ on S *with* $\theta(\psi) = \varphi$.

(7.5) **Lemma.** *We can choose an open covering $U_j (j \in J)$ of an open neighborhood U of E such that $E \cap U_j\colon x_j = 0$, $\varphi = w_j^{-1} dw_j$, $\psi = x_j^{-1} w_j^{-1} dw_j \wedge dw_j$, $\theta = x_j \partial/\partial x_j$ on U_j where x_j and $w_j (= W_{|U_j})$ are parameters on U_j.*

Proof. We choose a coordinate covering $U_j (j \in J)$ of a neighborhood U of E with parameters z_j and $w_j (= W_{|U_j})$. Indeed w_j is always part of parameters on a small neighborhood U in view of (7.3). We express first $\psi = f_j(z_j, w_j)\, z_j^{-1} w_j^{-1} dz_j \wedge dw_j$. Then f_j is by the choice of ψ a holomorphic function on U_j such that $f_j(0, w_j) = 1$. Then by a solution g_j of a differential equation $\partial g_j/\partial z_j = z_j^{-1}(f_j(z_j, w_j) - 1)$, we define $x_j = z_j \exp(g_j)$. Then x_j and w_j is a coordinate system on U_j by shrinking U_j if necessary, and $\psi = x_j^{-1} w_j^{-1} dx_j \wedge dw_j$. Then by definition $\theta = x_j \partial / \partial x_j$ on U_j. Q.E.D.

(7.6) **Lemma.** *The set of zeroes of the vector field θ is the union of E and singular points of Z. Moreover $\text{index}_p\, \theta = 1$ for any singular point p of Z.*

Proof. We denote by Zero (σ) the set of zeroes of a vector field σ. At a nonsingular point p of Z we have parameters x and $w\,(= W_{|U})$ and write $\varphi = w^{-1} dw$, $\psi = f(x, w)\, w^{-1}\, dx \wedge dw$ where f is a nonvanishing holomorphic function on U, a neighborhood of p. Then $\theta = f(x, w)^{-1}\, \partial / \partial x$ by definition. Hence in particular θ does not vanish at p. At a singular point p, we can write $W = xy$, $\varphi = x^{-1} dx + y^{-1} dy$, $\psi = f x^{-1} y^{-1} dx \wedge dy$ and $\theta = f^{-1} (x \partial / \partial x - y \partial / \partial y)$, with parameters x and y at p where f is a nonvanishing holomorphic function. Hence Zero (θ) contains Sing (Z). In view of (6.11) and (7.5) Zero $(\theta) = E \cup \text{Sing}(Z) \cup \{p_1, \cdots, p_N\}$, where p_i is an isolated zero of θ. Since E is a nonsingular elliptic curve and the local structure near E is isomorphic to that of the zero section of the normal bundle of E in S, we can find a differentiable vector field θ' on a small neighborhood U of E such that the one parameter group associated to θ' induces nontrivial translations of E. This means that $\theta' = a_1 \partial / \partial u_1 + a_3 \partial / \partial v_1 + a_4 \partial / \partial v_2$, where $z_j = \log w_j$, $u_1 =$ a linear combination of Re z_j and Im z_j, $v_1 = \text{Re}\, x_j$, $v_2 = \text{Im}\, x_j$, and $a_{1|E} =$ a nonvanishing function on E. So we let $\theta'' = \theta + h\theta'$ with a differentiable function h with support in U, $h \equiv 1$ on U', U' being an open set with $\bar{U}' \subset U$. Then Zero (θ'') $= (\text{Zero}(h) \cap \text{Zero}(\theta) \cap U) \quad \cup (\text{Zero}(\theta) \cap (S - U')) \quad = \text{Sing}(Z) \cup \{p_1, \cdots, p_N\}$, hence Zero (θ'') is isolated. Now we can apply a theorem of Hopf. We infer that the Euler number $\chi(S)$ of S is equal to $\sum_{p \in \text{Zero}(\theta'')} \text{index}_p(\theta'') \geqq n$ because $\text{index}_p(\theta'') = \text{index}_p(\theta) \geqq 1$ for an isolated zero point p. In view of (6.11) $\chi(S) = n$ whence $N = 0$, Zero $(\theta) = E \cup \text{Sing}(Z)$, $\text{index}_p(\theta) = 1$ for any $p \in \text{Sing}(Z)$. Q.E.D.

(7.7) **Lemma.** *There exist $\omega_j \in \Gamma(U_j, \Omega_S^1(\log(E+Z)))$ such that*

$$\psi = \omega_j \wedge \varphi, \quad d\omega_j = 0, \quad \omega_j = \omega_k + c_{jk}\varphi, \quad c_{jk} \in H^1(S, \mathbf{C})$$

for an open covering $\{U_j\}$ of S.

Proof. Let $\tilde{\Omega} = \Omega_S^1(\log(E+Z))$. First we shall show that there exists $\omega_j (\in \Gamma(U_j, \tilde{\Omega}))$ such that $\psi = \omega_j \wedge \varphi$. If U_j is contained in $S - Z$, then w_j (= the restriction of W to U_j) is part of local parameters in view of (7.6) and its proof. Since $(\psi) = -E - Z$ and w_j is nonvanishing on $S - Z$, we can write $\psi = \omega_j \wedge w_j^{-1} dw_j$ where ω_j is a meromorphic 1 form with logarithmic poles along $E \cap U_j$. At a nonsingular point of Z, $w_j (= W_{|U_j})$ is a defining equation of Z, so that w_j is part of local parameters. Therefore there exists a holomorphic 1 form ω_j such that $\psi = \omega_j \wedge w_j^{-1} dw_j$. Finally we consider the problem at a singular point p of Z. Let U_j be an open ball with parameters x and y with p: $(x, y) = (0, 0)$, $W = xy$. We can write

$$\psi = f(x, y)\, x^{-1} y^{-1}\, dx \wedge dy, \quad \varphi = x^{-1} dx + y^{-1} dy$$

with a nonvanishing holomorphic function f on U_j. Then $\omega_j = (1/2)\, f(x, y)\, (x^{-1} dx - y^{-1} dy)$ satisfies $\psi = \omega_j \wedge \varphi$. By the above choice ω_j is in $\Gamma(U_j, \tilde{\Omega})$. On $U_j \cap U_k$ we have $\omega_j = \omega_k + g_{jk}\varphi$. Since ω_j is in $\Gamma(U_j, \tilde{\Omega})$, g_{jk} is a one cocycle in $H^1(S, \mathcal{O}_S)$. By choosing a finer covering of U_j if necessary, we have holomorphic functions g_j and a one cocycle c_{jk} in $H^1(S, \mathbf{C})$ such that

$$g_{jk} = -g_j + g_k + c_{jk}, \quad \omega_j + g_j\varphi = \omega_k + (g_k + c_{jk})\,\varphi.$$

Let $\omega_j^* = \omega_j + g_j\varphi$. Then $d\omega_j^* = d\omega_k^*$ is a meromorphic 2 form with poles along $E+Z$. Since $K+[E+Z]=0$, $d\omega_j^*$ is a constant multiple of ψ, $d\omega_j^* = c\psi$. We shall show $c=0$. On a neighborhood U of a point p of E we can express $\psi = x^{-1}w^{-1}dx \wedge dw$ with suitable coordinates x and $w = W_{|U}$. Then $\omega_j^* = x^{-1}dx + h(x,w)\,w^{-1}dw$ for a holomorphic function h on U. Since $d\omega_j^* = h_x(x,w)\,w^{-1}dx \wedge dw$, we have $h_x(x,w) = c/x$ which shows $c=0$. Q.E.D.

(7.8) **Lemma.** $\pi_1(S-E) \cong \pi_1(S) \cong \mathbf{Z}$.

Proof. Let $\pi\colon \mathcal{S} \to D$ be a one-parameter family in (5.11). Then it is clear that $\pi_1(\mathcal{S}_t - \mathcal{E}_t) \cong \pi_1(\mathcal{S}_0 - \mathcal{E}_0) = \pi_1(S-E)$. Since $\mathcal{S}_t$ is a primary Hopf surface blown-up with centers on the image $(\mathcal{E}_1)_t$ of $\mathcal{E}_t$, the natural homomorphism π_1 (a primary Hopf surface $-(\mathcal{E}_1)_t) \to \pi_1(\mathcal{S}_t - \mathcal{E}_t)$ is onto. Hence $\pi_1(\mathcal{S}_t - \mathcal{E}_t)$ is abelian, therefore $\pi_1(S-E)$ is abelian. In view of (6.17) $\pi_1(S-E) \cong H_1(S-E, \mathbf{Z}) \cong H_1(S, \mathbf{Z})\ (\cong \pi_1(S)) \cong \mathbf{Z}$. Q.E.D.

(7.9) **Lemma.** *There exist a single-valued holomorphic function W and a single-valued meromorphic function X on $\tilde{S}$, holomorphic at $\pi^{-1}(E)$, such that*

$$g^*W = \beta W, \quad g^*X = W^n X,$$

$\psi =$ a constant multiple of $X^{-1}W^{-1}dX \wedge dW$, $\varphi = W^{-1}dW$.

Proof. Number the irreducible components $\tilde{Z}_\lambda$ of $\pi^{-1}(Z)$ consecutively so that $g(\tilde{Z}_\lambda) = \tilde{Z}_{\lambda+n}\,(\lambda \in \mathbf{Z})$. In view of (7.7) there exist meromorphic 1 forms ω_j on U_j with logarithmic poles along $E+Z$ such that $\psi = \omega_j \wedge \varphi$, $d\omega_j = 0$, $\omega_j = \omega_k + c_{jk}\varphi$, $\{c_{jk}\} \in H^1(S, \mathbf{C})$. Therefore we have an open covering $\{U_{j\lambda}\}$ of $\tilde{S}$ with $\pi^{-1}(U_j) = \bigcup_{\lambda \in \mathbf{Z}} U_{j\lambda}$, $g(U_{j\lambda}) = U_{j\lambda+1}$ and $c_{j\lambda} \in \mathbf{C}$ such that $c_{jk} = -c_{j\lambda} + c_{k\nu}$ for $U_{j\lambda} \cap U_{k\nu} \neq \phi$. Therefore $\omega := \pi^*\omega_j + c_{j\lambda}\pi^*\varphi$ is a d-closed meromorphic 1-form on $\tilde{S}$ with logarithmic poles along $\pi^{-1}(E+Z)$. Since $K+[E+Z]$ restricts to the dualising sheaf ω_Z of $Z\,(\cong \mathcal{O}_Z)$ $\mathrm{Res}_Z\psi$ is a nonzero multiple of a generator ω_0 of $H^0(Z, \omega_Z)$ with $\mathrm{Res}_p\omega_0 = \pm 1$ for any singular point p of Z, say, $\mathrm{Res}_Z\psi = \omega_0/a$. Let p_k be the intersection point of $\tilde{Z}_k$ and $\tilde{Z}_{k+1}$. Then we may suppose the $\mathrm{Res}_{p_k}(\omega_0)_{\tilde{Z}_k} = +1$, $\mathrm{Res}_{p_k}(\omega_0)_{\tilde{Z}_{k+1}} = -1$ for any k. Writing $\psi = f(x,y)\,x^{-1}y^{-1}dx \wedge dy$ on a neighborhood V of p_k, we have $a^{-1} = f(0,0)$ by the proof of (7.7). Let $\tilde{\psi} = a\psi$, $\tilde{\omega} = a\omega$,

$$\omega(k) = \tilde{\omega} - (\mathrm{Res}_{\tilde{Z}_k}\tilde{\omega})\pi^*\varphi, \quad W = \exp\left(\int_o^x \pi^*\varphi\right), \quad X_k = \exp\left(\int_o^x \omega(k)\right), \quad Y_k = X_{k+1}^{-1}(k \in \mathbf{Z}),$$

$X = X_0$ where o is a point of $\tilde{S} - \pi^{-1}(E+Z)$. Then we have $\mathrm{Res}_{\tilde{Z}_{k+1}}\tilde{\omega} - \mathrm{Res}_{\tilde{Z}_k}\tilde{\omega} = a(\mathrm{Res}_{\tilde{Z}_{k+1}}\omega - \mathrm{Res}_{\tilde{Z}_k}\omega) = af(0,0) = 1$. Hence $X_kY_k = W$. Since X_k and Y_k are singlevalued meromorphic on neighborhoods of $\tilde{Z}_k$ and $\tilde{Z}_{k+1}$ respectively, they are meromorphic on the union of the neighborhoods. Repeating this argument, it can be shown that X_k and Y_k are singlevalued meromorphic on a neighborhood of $\pi^{-1}(Z)$. Hence X_k and Y_k are (possibly multivalued) meromorphic on $\tilde{S} - \pi^{-1}(E)$.

In view of (7.8), $\pi_1(S-E) \cong \pi_1(S) \cong \mathbf{Z}$, hence $\pi_1(\tilde{S}-\pi^{-1}(E)) = \{1\}$. It follows that X_k and Y_k are singlevalued meromorphic on $\tilde{S}-\pi^{-1}(E)$. We have $X_k = W^{-k}X$, $Y_k = W^{k+1}X^{-1}$, $X_{-n} = W^n X$. Next we shall show that

$$g^*X = cW^nX, \quad c = \exp\left(\int_o^{g\cdot o} \omega(0)\right).$$

In fact,

$$g^*X = \exp\left(\int_o^{g\cdot x} \omega(0)\right) = c\cdot\exp\left(\int_{g\cdot o}^{g\cdot x} \omega(0)\right) = c\cdot\exp\left(\int_o^x g^*\omega(0)\right)$$
$$= c\cdot\exp\left((c_{j\lambda+1}-c_{j\lambda})\int_o^x \pi^*\varphi\right)\exp\left(\int_o^x \omega(0)\right).$$

Hence $g^*X = cW^{m-n}X_{-n}$, $m = c_{j\lambda+1}-c_{j\lambda}$. Both g^*X and X_{-n} are nonvanishing at a generic point of $\tilde{Z}_{-n}$ so that $n = m$ and $g^*X = cW^nX$. By multiplying W by $c^{1/n}$, we have $g^*W = W^nX$. On a sufficiently small open neighborhood $U = \bigcup_j U_j$ of E, $\psi = x_j^{-1}W^{-1}dx_j \wedge dW$, $\varphi = W^{-1}dW$ by (7.5). Since U is isomorphic to a neighborhood of the zero section of the normal bundle $N_{E/S}$ of E, we have projections p of U onto E and $\tilde{p}$ of $\pi^{-1}(U)$ onto $\pi^{-1}(E)$ with $\pi\tilde{p} = p\pi$. We may suppose that $V_j = p(U_j)$, $U_j = p^{-1}(V_j)$, $\{V_j\}$ is a covering of E. $E\cap U_j$ is defined by $x_j = 0$ in U_j and there exist $f_{jk}(W) \in H^0(V_j\cap V_k, \mathcal{O}_E^*)$ such that $x_j = p^*f_{jk}(W)x_k$. Since $\pi^*[E]_E$ is trivial, we have a covering $V_{j\lambda}(\lambda\in\mathbf{Z})$ of $\pi^{-1}(V_j)$ and a nonvanishing holomorphic function $f_{j\lambda}$ on $V_{j\lambda}$ such that $\pi^*f_{jk} = f_{j\lambda}^{-1}f_{k\nu}$. We define $u = x_j\tilde{p}^*f_{j\lambda}$. Then u is a holomorphic function on $\pi^{-1}(U)$ and we have $\psi = u^{-1}W^{-1}du \wedge dW$, $X^{-1}dX = au^{-1}du + h(u, W)dW$ on $\pi^{-1}(U-E)$ for some h. It follows that $h(u, W)$ is independent of u, so we may write $h(u, W) = h(W)$. So $X = u^a\exp(\int h(W)dW)$. So by defining $v = u\cdot\exp(\int h(W)dW/a)$, *we have* $X = v^a$. Since X is singlevalued on $\pi^{-1}(S-E)$, a is an integer. Since $g^*v = W^{n/i}v$, v is a defining equation of $\pi^{-1}(E)$ in $\tilde{S}$, and $g^*W = \beta W$, $0 < |\beta| < 1$, we infer that a is a positive integer. Thus X is holomorphic at $\pi^{-1}(E)$ and $\tilde{\psi} = a\psi = x^{-1}w^{-1}dX \wedge dW$. Q.E.D.

(7.10) Now we are in a position to prove that S *is isomorphic to a parabolic Inoue surface* $S(\beta, n)$, $n = b_2(S)$.

Proof. We keep the same notations $g_n(t)$, $\mathcal{X}$ etc. as in §1. Let $X_k := W^{-k}X$, $Y_k := WX_k^{-1} = X_{k+1}^{-1}$, $U_k := \tilde{S}-\pi^{-1}(E)\cup\left(\bigcup_{\lambda\neq k, k+1}\tilde{Z}_\lambda\right)$, $V := \tilde{S}-\pi^{-1}(Z)$ where X is the meromorphic function in (7.9). Then X_k and Y_k are singlevalued on U_k by (7.9). Moreover they are *holomorphic* on U_k. Indeed, in view of (7.9), X_k and Y_k are holomorphic on $\tilde{S}-\pi^{-1}(E\cup Z)$, $\mathrm{Res}_{\tilde{Z}_k}\omega(k) = 0$ so that X_k is nonvanishing holomorphic at $\tilde{Z}_k - \tilde{Z}_{k+1}\cup\tilde{Z}_{k-1}$. Hence Y_k is holomorphic there. Since X_{k+1} is nonvanishing at $\tilde{Z}_{k+1} - \tilde{Z}_k\cup\tilde{Z}_{k+2}$, $X_k = WX_{k+1}$ and $Y_k = X_{k+1}^{-1}$ are holomorphic there. Therefore X_k and Y_k are holomorphic on a neighborhood of $p_k = \tilde{Z}_k\cdot\tilde{Z}_{k+1}$, hence on U_k by a theorem of Hartogs. On the other hand $X = X_0$ is holomorphic on V by (7.9). We define a mapping of $\tilde{S}$ into $\mathcal{X}$ by

$$\tilde{f}(U_k)\subset\mathcal{U}_k, \quad \tilde{f}(V)\subset\mathcal{V},$$
$$\tilde{f}^*x_k = X_k, \quad \tilde{f}^*y_k = Y_k, \quad \tilde{f}^*w = W, \quad \tilde{f}^*x = X.$$

From the above consideration $\tilde{f}$ is holomorphic. Moreover $\tilde{f}\cdot g = g_n(\beta^{1/n})\circ \tilde{f}$ whence we have a holomorphic mapping f of S into $S(\beta, n)$. It is evident that f is of maximal rank on a neighborhood of Z. Since $f^*(x^{-1}w^{-1}dx\wedge dw) = \tilde{\psi}$, f is of maximal rank on $S - E\cup Z$. Since f is proper, f is surjective. Since f is bijective on Z, $\deg f$ is equal to one. Consequently f is an isomorphism of S onto $S(\beta, n)$. Q.E.D.

This completes the first proof of (7.1).

(7.11) **Lemma.** *Let S be a* VII_0 *surface with two cycles A and B of rational curves. Then* $\pi_1(S)\cong\pi_1(S-A)\cong\pi_1(S-B)\cong\mathbf{Z}$.

Proof. By (4.8) there exists a one-parameter smooth proper family $\pi\colon \mathscr{S}\to D$ with two divisors $\mathscr{A}$ and $\mathscr{B}$ flat over D such that $(\mathscr{S},\mathscr{A},\mathscr{B})_0\cong(S,A,B)$, $\mathscr{A}_t$ is a smooth elliptic curve for $t\neq 0$, and $\mathscr{B}_t\cong B$ for any t. Then in view of (7.1) $\mathscr{S}_t$ is a blown-up parabolic Inoue surface, $\mathscr{B}_t$ is a proper transform of a cycle of rational curves. In view of (1.1) any parabolic Inoue surface minus a cycle of rational curves has a commutative fundamental group $\mathbf{Z}^2$. Therefore we have a natural homomorphism of $\mathbf{Z}^2$ onto $\pi_1(\mathscr{S}_t-\mathscr{B}_t)$. Therefore $\pi_1(S-B)\cong\pi_1(\mathscr{S}_t-\mathscr{B}_t)$ is abelian. Hence $\pi_1(S-B)\cong H_1(S-B,\mathbf{Z})\cong H_1(S,\mathbf{Z})\cong\pi_1(S)\cong\mathbf{Z}$ by (6.10). Similarly $\pi_1(S-A)\cong\pi_1(S)\cong\mathbf{Z}$. Q.E.D.

(7.12) **Adderndum.** Here we shall give another proof of (7.1) by applying [2]. Let S be a VII_0 surface with an elliptic curve E and a cycle Z of rational curves. Then by (2.12) we have $Z^2=0$. Hence by the main theorem of [2] S is isomorphic to $S_{n,\beta,t}$ for some n, β and t where $n=-E^2$, $0<|\beta|<1$, $t=(t_1,\cdots,t_n)\in\mathbf{C}^n$. (See [2] for the definition of $S_{n,\beta,t}$.) In view of [2, Prop. 7.1] $S-Z$ is an affine $\mathbf{C}$-bundle $A_{n,\beta,t}$ over an elliptic curve E'. The curve E' is defined as follows. Let $\tilde{S}$ be the universal covering of S, g a transformation of $\tilde{S}$ which corresponds to a generator of $\pi_1(S)$, W the function on $\tilde{S}$ defined in (7.4). There is a constant β such that $g^*W=\beta W$, $0<|\beta|<1$, by taking g^{-1} instead of g if necessary. Then E' is a quotient of $\mathbf{C}^*$ by the multiplication by β. Let Σ be a natural compactification of $A_{n,\beta,t}$ as a $\mathbf{P}_1$ bundle over E'. Let F be a fiber of it, Γ the curve $\Sigma - A_{n,\beta,t}(\cong E')$. By [2, Prop. 7.1] $\Gamma^2=n$. Since $EZ=0$ and $S-Z$ is (viewed as) an open subset of Σ, E is a curve on Σ. Since $H_2(\Sigma,\mathbf{Z})$ is generated by F and Γ, we write $E=a\Gamma+bF$. By $E^2=-n$, $E\Gamma=0$, $EF\geqq 0$, we have $a=1$, $b=-n$. This shows that E is a section of the $\mathbf{P}_1$ bundle, hence of the bundle $A_{n,\beta,t}$, and E' is isomorphic to E. So $t_1=\cdots=t_n=0$, S is isomorphic to $S_{n,\beta,0}$. By definition $S_{n,\beta,0}$ is a parabolic Inoue surface $S(\beta,n)$. This completes the second proof of (7.1). Q.E.D.

§8. Hyperbolic Inoue surfaces

The purpose of this section is to prove the following

(8.1) **Theorem.** *Let S be a* VII_0 *surface with two cycles of rational curves. Then S is isomorphic to a hyperbolic Inoue surface* $S_{\omega}^{[n]}$.

(8.2) Let S be a VII_0 surface with two cycles A and B of rational curves. Then in view of (5.4) and (2.15) $\pi_1(S)\cong H_1(S,\mathbf{Z})\cong H_1(A,\mathbf{Z})\cong H_1(B,\mathbf{Z})$. Let $\pi\colon\tilde{S}\to S$ be the universal covering of S. Then $\pi^{-1}(A)$ and $\pi^{-1}(B)$ consist of infinite chains C and D of nonsingular rational curves with $C_\lambda C_{\lambda+1} = D_\nu D_{\nu+1} = 1$, $C_\lambda C_\nu = D_\mu D_\delta = 0$ $(\lambda - \nu, \mu - \delta \neq 0, \pm 1)$ where $C = \sum_{\lambda \in \mathbf{Z}} C_\lambda$, $D = \sum_{\nu \in \mathbf{Z}} D_\nu$. Let $n_\lambda = -C_\lambda^2$, $m_\nu = -D_\nu^2$. Let g be a covering transformation of π which generates $H_1(S, \mathbf{Z})$. We may assume that $g(C_\lambda) = C_{\lambda - r}$, $g(D_\nu) = D_{\nu - s}$. Let $A_{\lambda \bmod r} = \pi(C_\lambda)$, $B_{\nu \bmod s} = \pi(D_\nu)$, $A = \pi(C)$, $B = \pi(D)$. Then we have $A_\lambda^2 = -n_\lambda (r > 1)$, $A_0^2 = -n_0 + 2\,(r = 1)$, $B_\nu^2 = -m_\nu (s > 1)$ and $B_0^2 = -m_0 + 2\,(s = 1)$. In view of (3.11) and (6.1) we have $n(S) = 2$. Let F_1 and F_2 be flat line bundles in $N(S)$, ω_λ be a nonzero section of $H^0(S, \Omega_S^1(\log(A+B))(F_\lambda))$ $(\lambda = 1, 2)$. Then by the definition of $\pi: \tilde{S} \to S$, the pull backs $\pi^* F_1$ and $\pi^* F_2$ are trivial on $\tilde{S}$. By fixing their trivializations, $\pi^* \omega_1$ and $\pi^* \omega_2$ are naturally viewed as global sections of $\Omega_{\tilde{S}}^1(\log \pi^*(A+B))$ which we denote by $\tilde{\omega}_1$ and $\tilde{\omega}_2$ respectively. By the isomorphism $H^1(S, \mathbf{C}^*) \cong \operatorname{Hom}(H_1(S, \mathbf{Z}), \mathbf{C}^*)$, there corresponds a character χ of $H_1(S, Z)$ to the line bundle F_1. Then χ^{-1} corresponds to F_2 by (3.8). Let $\sigma = \chi(g)$. Then we have $g^* \tilde{\omega}_1 = \sigma \tilde{\omega}_1$, $g^* \tilde{\omega}_2 = \sigma^{-1} \tilde{\omega}_2$.

(8.3) **Lemma.** *Let U be a disc $U := \{(x_1, x_2); |x_\nu| < \varepsilon\}$, D_μ a divisor defined by $x_\mu = 0$. Let ω_1 and ω_2 be d-closed meromorphic* 1 *forms with logarithmic poles along $D_1 + D_2$ with $\omega_1 \wedge \omega_2 \neq 0$. Assume moreover that the residues* $R_{\nu\mu} = \operatorname{Res}_{D_\mu} \omega_\nu$ *are constants. Let* $R = (R_{\nu\mu})$, *and let $(\omega_1 \wedge \omega_2)$ be the divisor defined by $\omega_1 \wedge \omega_2$. If $(\omega_1 \wedge \omega_2) = -D_1 - D_2$, then* $\det R \neq 0$ *and we have a system y_1, y_2 of parameters on U such that*

$$\omega_\nu = R_{\nu 1} y_1^{-1} dy_1 + R_{\nu 2} y_2^{-1} dy_2, \quad D_\mu : y_\mu = 0.$$

Proof. By assumption we write $\omega_\nu = F_{\nu 1} x_1^{-1} dx_1 + F_{\nu 2} x_2^{-1} dx_2$ for $F_{\nu\mu}$ holomorphic so that $\omega_1 \wedge \omega_2 = \det(F_{\nu\mu}) x_1^{-1} x_2^{-1} dx_1 \wedge dx_2$. This shows $\det(F_{\nu\mu}) \neq 0$ hence $\det R \neq 0$. Letting

$$\omega'_\nu = R_{\nu 1} x_1^{-1} dx_1 + R_{\nu 2} x_2^{-1} dx_2,$$

we have d-closed holomorphic 1-forms $\omega_\nu - \omega'_\nu$ which we can write

$$\omega_\nu - \omega'_\nu = dg_\nu$$

for holomorphic functions g_ν. Letting $(R^{\nu\mu})$ be the inverse matrix of R, $y_\nu = x_\nu \exp(R^{\nu 1} g_1 + R^{\nu 2} g_2)$, we have

$$\omega_\nu = R_{\nu 1} y_1^{-1} dy_1 + R_{\nu 2} y_2^{-1} dy_2 \quad \text{and} \quad D_\nu : y_\nu = 0. \quad \text{Q.E.D.}$$

(8.4) **Lemma.** *There exists a system x_λ and y_λ of parameters on an open neighborhood U_λ of $C_{\lambda-1} \cup C_\lambda - C_{\lambda+1} - C_{\lambda-2}$ such that*

$$C_\lambda : x_\lambda = 0, \quad C_{\lambda-1} : y_\lambda = 0,$$

$$\tilde{\omega}_1 = a_\lambda y_\lambda^{-1} dy_\lambda + b_\lambda x_\lambda^{-1} dx_\lambda, \quad \tilde{\omega}_2 = c_\lambda y_\lambda^{-1} dy_\lambda + d_\lambda x_\lambda^{-1} dx_\lambda$$

where a_λ, b_λ, c_λ and d_λ are constants with $a_\lambda d_\lambda - b_\lambda c_\lambda \neq 0$.

Proof. Since ω_1 and ω_2 are meromorphic 1-forms on S with logarithmic poles, so are $\tilde{\omega}_1$ and $\tilde{\omega}_2$ on $\tilde{S}$. Hence $\operatorname{Res}_{C_\lambda} \tilde{\omega}_1$ and $\operatorname{Res}_{C_\lambda} \tilde{\omega}_2$ are global sections of $\mathcal{O}_{C_\lambda}$ so that they are constants. By (2.8), we have $(\tilde{\omega}_1 \wedge \tilde{\omega}_2) = \pi^*(\omega_1 \wedge \omega_2) = -\pi^*(A+B) = -C - D$. Since $C_\lambda^2 < 0$, we have a Stein neighborhood U of $C_\lambda - C_{\lambda+1}$ with parameters x and y such that $C_\lambda \cap U: x=0$, $C_{\lambda-1} \cap U: y=0$. By (8.3) we have a system x_λ and y_λ of parameters on U such that

$$C_\lambda \cap U: x_\lambda = 0, \quad C_{\lambda-1} \cap U: y_\lambda = 0,$$
$$\tilde{\omega}_1 = a_\lambda y_\lambda^{-1} dy_\lambda + b_\lambda x_\lambda^{-1} dx_\lambda, \quad \tilde{\omega}_2 = c_\lambda y_\lambda^{-1} dy_\lambda + d_\lambda x_\lambda^{-1} dx_\lambda.$$

Since $C_{\lambda-1}^2 < 0$, there exists a Stein open neighborhood V of $C_{\lambda-1} - C_{\lambda-2}$ with parameters u' and v' such that $C_\lambda \cap V: u'=0$, $C_{\lambda-1} \cap V: v'=0$. By the same proof we can construct a system u and v of parameters on V such that

$$C_\lambda \cap V: u=0, \quad C_{\lambda-1} \cap V: v=0,$$
$$\tilde{\omega}_1 = a_\lambda v^{-1} dv + b_\lambda u^{-1} du, \quad \tilde{\omega}_2 = c_\lambda v^{-1} dv + d_\lambda u^{-1} du.$$

Consequently x_λ and y_λ are constant multiples of u and v respectively so that they are holomorphic on V. Letting $U_\lambda = U \cup V$, we have our lemma. Q.E.D.

(8.5) **Lemma.** *There exists a system z_ν and w_ν of parameters on a neighborhood V_ν of $D_\nu \cup D_{\nu-1} - D_{\nu+1} - D_{\nu-2}$ such that*

$$D_\nu: z_\nu = 0, \quad D_{\nu-1}: w_\nu = 0,$$
$$\tilde{\omega}_1 = a_\nu^* w_\nu^{-1} dw_\nu + b_\nu^* z_\nu^{-1} dz_\nu, \quad \tilde{\omega}_2 = c_\nu^* w_\nu^{-1} dw_\nu + d_\nu^* z_\nu^{-1} dz_\nu.$$

The proof is the same as in (8.4).

(8.6) **Lemma.** *By modifying by constant multiples, we have*

$$x_{\lambda+1} = y_\lambda^{-1}, \quad y_{\lambda+1} = x_\lambda y_\lambda^{n_\lambda},$$
$$z_{\nu+1} = w_\nu^{-1}, \quad w_{\nu+1} = z_\nu w_\nu^{m_\nu}.$$

Moreover $a_\lambda d_\lambda - b_\lambda c_\lambda = a_0 d_0 - b_0 c_0 \neq 0$, $a_\nu^* d_\nu^* - b_\nu^* c_\nu^* = a_0^* d_0^* - b_0^* c_0^* \neq 0$ *for any λ and ν.*

Proof. It suffices to prove the assertion about x_λ and y_λ. By (8.4)

$$\tilde{\omega}_1 = a_\lambda y_\lambda^{-1} dy_\lambda + b_\lambda x_\lambda^{-1} dx_\lambda = a_{\lambda+1} y_{\lambda+1}^{-1} dy_{\lambda+1} + b_{\lambda+1} x_{\lambda+1}^{-1} dx_{\lambda+1},$$
$$\tilde{\omega}_2 = c_\lambda y_\lambda^{-1} dy_\lambda + d_\lambda x_\lambda^{-1} dx_\lambda = c_{\lambda+1} y_{\lambda+1}^{-1} dy_{\lambda+1} + d_{\lambda+1} x_{\lambda+1}^{-1} dx_{\lambda+1}.$$

We have $\mathrm{Res}_{C_\lambda} \tilde{\omega}_1 = a_{\lambda+1} = b_\lambda$, $\mathrm{Res}_{C_\lambda} \tilde{\omega}_2 = c_{\lambda+1} = d_\lambda$. Therefore

$$(a_\lambda d_\lambda - b_\lambda c_\lambda) y_\lambda^{-1} dy_\lambda = -(a_{\lambda+1} d_{\lambda+1} - b_{\lambda+1} c_{\lambda+1}) x_{\lambda+1}^{-1} dx_{\lambda+1}.$$

Consequently by modifying by constant multiples, we infer

$$x_{\lambda+1} = y_\lambda^a, \quad y_{\lambda+1} = x_\lambda y_\lambda^b$$

for some constants a and b. Since $U_\lambda \cap U_{\lambda+1}$ constains a neighborhood of $C_\lambda - C_{\lambda-1} \cup C_{\lambda+1}$, we can continue y_λ analytically along a path going once around $C_{\lambda-1}$. By this analytic continuation of y_λ, we have the same relations among x_λ, y_λ, $x_{\lambda+1}$ and $y_{\lambda+1}$. Therefore $a = \pm 1$ and b is an integer. If $a=1$, then $dx_{\lambda+1} = dy_\lambda$ is a nontrivial holomorphic 1-form on C_λ which contradicts that C_λ is a rational curve. Consequently $a=-1$. It is easy to see that $b = -C_\lambda^2 = n_\lambda$. From $a=-1$ it follows that $a_\lambda d_\lambda - b_\lambda c_\lambda = a_{\lambda+1} d_{\lambda+1} - b_{\lambda+1} c_{\lambda+1} \neq 0$. Q.E.D.

(8.7) Let $\rho = a_0 d_0 - b_0 c_0$, $\rho^* = a_0^* d_0^* - b_0^* c_0^*$. Then by (8.6) we have

$$\begin{aligned} y_\lambda^{-1} dy_\lambda &= \rho^{-1}(d_\lambda \tilde{\omega}_1 - b_\lambda \tilde{\omega}_2), \quad & x_\lambda^{-1} dx_\lambda &= \rho^{-1}(-c_\lambda \tilde{\omega}_1 + a_\lambda \tilde{\omega}_2), \\ w_\nu^{-1} dw_\nu &= \rho^{*-1}(d_\nu^* \tilde{\omega}_1 - b_\nu^* \tilde{\omega}_2), \quad & z_\nu^{-1} dz_\nu &= \rho^{*-1}(-c_\nu^* \tilde{\omega}_1 + a_\nu^* \tilde{\omega}_2). \end{aligned} \tag{8.7.1}$$

So we define

$$\begin{aligned} Y_\lambda = Y_\lambda(P) &= \exp\left(\rho^{-1} \int_o^P (d_\lambda \tilde{\omega}_1 - b_\lambda \tilde{\omega}_2)\right), \\ X_\lambda = X_\lambda(P) &= \exp\left(\rho^{-1} \int_o^P (-c_\lambda \tilde{\omega}_1 + a_\lambda \tilde{\omega}_2)\right), \\ W_\nu = W_\nu(P) &= \exp\left(\rho^{*-1} \int_o^P (d_\nu^* \tilde{\omega}_1 - b_\nu^* \tilde{\omega}_2)\right), \\ Z_\nu = Z_\nu(P) &= \exp\left(\rho^{*-1} \int_o^P (-c_\nu^* \tilde{\omega}_1 + a_\nu^* \tilde{\omega}_2)\right) \end{aligned} \tag{8.7.2}$$

where o is the base point, o and $P \in \tilde{S} - C - D$.

It is clear from definition that

$$\begin{aligned} \tilde{\omega}_1 &= a_\lambda Y_\lambda^{-1} dY_\lambda + b_\lambda X_\lambda^{-1} dX_\lambda = a_\nu^* W_\nu^{-1} dW_\nu + b_\nu^* Z_\nu^{-1} dZ_\nu, \\ \tilde{\omega}_2 &= c_\lambda Y_\lambda^{-1} dY_\lambda + d_\lambda X_\lambda^{-1} dX_\lambda = c_\nu^* W_\nu^{-1} dW_\nu + d_\nu^* Z_\nu^{-1} dZ_\nu. \end{aligned} \tag{8.7.3}$$

(8.8) **Lemma.** *X_λ, Y_λ, Z_ν and W_ν are single-valued meromorphic functions on $\tilde{S}$ related by,*

$$\begin{aligned} X_{\lambda+1} &= Y_\lambda^{-1}, & Y_{\lambda+1} &= X_\lambda Y_\lambda^{n_\lambda}, \\ Z_{\nu+1} &= W_\nu^{-1}, & W_{\nu+1} &= Z_\nu W_\nu^{m_\nu}, \\ W_0 &= Y_0^e X_0^f, & Z_0 &= Y_0^g X_0^h \end{aligned}$$

where e, f, g and h are integers with $eh - fg = \pm 1$.

Proof. By (7.11), we have $\pi_1(S - B) \cong \pi_1(S) \cong H_1(S, \mathbf{Z})$. Hence $\tilde{S} - D$ is simply connected. Since $\varrho^{-1}(d_\lambda \tilde{\omega}_1 - b_\lambda \tilde{\omega}_2)$ and $\varrho^{-1}(-c_\lambda \tilde{\omega}_1 + a_\lambda \tilde{\omega}_2)$ have integral residues along C by (8.6), X_λ and Y_λ are therefore single-valued holomorphic on $\tilde{S} - C - D$, meromorphic on $\tilde{S} - D$. Similarly Z_ν and W_ν are single-valued and meromorphic on $\tilde{S} - C$. Comparing two expressions of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ in terms of X_0, Y_0, Z_0 and W_0, and modifying X_0 and Y_0 by constant multiples, we have

$$W_0 = Y_0^e X_0^f, \quad Z_0 = Y_0^g X_0^h$$

where e, f, g and h are integers. It is also possible to express X_0 and Y_0 in terms of Z_0 and W_0 so that $eh - fg = \pm 1$. Q.E.D.

(8.9) Let

$$\begin{aligned} N &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} n_{r-1} & 1 \\ -1 & 0 \end{bmatrix} \cdots \begin{bmatrix} n_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} n_0 & 1 \\ -1 & 0 \end{bmatrix}, \\ N^* &= \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix} = \begin{bmatrix} n_{s-1}^* & 1 \\ -1 & 0 \end{bmatrix} \cdots \begin{bmatrix} n_1^* & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} n_0^* & 1 \\ -1 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} e & f \\ g & h \end{bmatrix}. \end{aligned}$$

It is easy to check by using (2.12) that $Tr\,N > 2$, $Tr\,N^* > 2$. From $g(C_\lambda) = C_{\lambda-r}$, $g(D_\nu) = D_{\nu-s}$ it follows that $g^*X_\lambda = X_{\lambda+r}$, $g^*Y_\lambda = Y_{\lambda+r}$, $g^*Z_\nu = Z_{\nu+s}$ and $g^*W_\nu = W_{\nu+s}$ up to constant multiples. Therefore there exist constants $\hat{A}$ and $\hat{B}$ such that

$$g^*Y_0 = \hat{A}Y_r = \hat{A}Y_0^aX_0^b, \quad g^*X_0 = \hat{B}X_r = \hat{B}Y_0^cX_0^d.$$

Since $\operatorname{Tr} N > 2$, we can choose constants A_0 and B_0 satisfying

$$A_0^{a-1}B_0^b = \hat{A}, \quad A_0^cB_0^{d-1} = \hat{B}.$$

Then by replacing Y_0 and X_0 by A_0Y_0 and B_0X_0, we have

$$g^*Y_0 = Y_0^aX_0^b, \quad g^*X_0 = Y_0^cX_0^d.$$

Defining X_λ, Y_λ, Z_ν and W_ν by the relations in (8.8) again we obtain the following

(8.10) **Lemma.** *There exist single-valued meromorphic functions on* $\tilde{S}$ X_λ, Y_λ, Z_ν *and* W_ν *such that the relations* (8.7.3) and (8.8) *hold and*

$$\begin{aligned} g^*Y_0 &= Y_0^aX_0^b, & g^*X_0 &= Y_0^cX_0^d \\ g^*W_0 &= W_0^{a^*}Z_0^{b^*}, & g^*Z_0 &= W_0^{c^*}Z_0^{d^*}. \end{aligned}$$

(8.11) Let $\alpha\,(>1)$ and α' (= the conjugate of α over $\mathbf{Q}$) be the eigenvalues of N. Since $BN = N^*B$, they are also eigenvalues of N^*. Let $(\omega, 1)$ and $(\omega^*, 1)$ be eigenvectors of N and N^* such that

$$(\omega, 1)N = \alpha(\omega, 1), \quad (\omega^*, 1)N^* = \alpha(\omega^*, 1).$$

Therefore $(\omega^*, 1)BN = (\omega^*, 1)N^*B = \alpha(\omega^*, 1)B$, that is, $(\omega^*, 1)B$ is an eigenvector of N so that there exists β such that

$$(\omega^*, 1)B = \beta(\omega, 1).$$

Equivalently $e\omega^* + g = \beta\omega$, $f\omega^* + h = \beta$.
This implies that

$$\text{(8.11.1)} \quad \begin{aligned} \varphi_1 &= \omega^*W_0^{-1}dW_0 + Z_0^{-1}dZ_0 = \beta(\omega Y_0^{-1}dY_0 + X_0^{-1}dX_0), \\ \phi_2 &= (\omega^*)'W_0^{-1}dW_0 + Z_0^{-1}dZ_0 = \beta'(\omega' Y_0^{-1}dY_0 + X_0^{-1}dX_0) \end{aligned}$$

are eigenvectors of g^*. The transformation g induces a linear transformation g^* of $V = \mathbf{C}Y_0^{-1}dY_0 + \mathbf{C}X_0^{-1}dX_0$. Then $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are eigenvectors of g^* so that they are constant multiples of φ_1 and φ_2. We may assume that

$$\varphi_1 = \tilde{\omega}_1, \quad \varphi_2 = \tilde{\omega}_2, \quad \alpha = \sigma, \quad \alpha' = \sigma^{-1}.$$

(See (8.2) for the definition of σ.)

(8.12) **Lemma.** $\beta\beta' < 0$, $\det B = -1$.

Proof. Denoting $t = \operatorname{Tr} N = \operatorname{Tr} N^*$, we have

$$\begin{gathered} \alpha = (t + \sqrt{t^2 - 4})/2, \quad \omega = c(\alpha - a)^{-1}, \quad \omega^* = c^*(\alpha - a^*)^{-1}, \\ \omega - \omega' = b^{-1}\sqrt{t^2 - 4}, \quad \omega^* - \omega^{*\prime} = b^{*-1}\sqrt{t^2 - 4}. \end{gathered}$$

It is easy to check that b, $b^* > 0$. Hence $\omega - \omega' > 0$, $\omega^* - \omega^{*\prime} > 0$. By (8.11) $\det B = (\omega^* - \omega^{*\prime})^{-1}(\omega - \omega')\beta\beta'$. So if we prove $\beta\beta' < 0$, then $\det B = -1$ by (8.8).

Let

$$p = p(P) = |Y_0^{\omega} X_0|, \qquad q = q(P) = |Y_0^{\omega'} X_0|,$$
$$r = r(P) = |W_0^{\omega *} Z_0|, \quad s = s(P) = |W_0^{\omega *'} Z_0|.$$

By (8.11) $p^{\beta} = r$, $q^{\beta'} = s$, $g^* p = p^{\alpha}$, $g^* q = q^{\alpha'}$, $g^* r = r^{\alpha}$, $g^* s = s^{\alpha'}$. Then p and q (resp. r and s) are continuous on $\tilde{S} - D$ (resp. $\tilde{S} - C$) (cf. [6, p. 96, 97]). We shall derive a contradiction by assuming $\beta\beta' > 0$. First we assume that $\beta > 0$, $\beta' > 0$. Then we have

$$p(P) < 1 \quad \text{iff} \quad r(P) < 1,$$
$$q(P) < 1 \quad \text{iff} \quad s(P) < 1.$$

Clearly $p(P) < 1$ iff $g^* p(P) = p(g(P)) < 1$, etc. We define g invariant closed subsets S_k $(k = 1, 2, 3, 4)$ of $\tilde{S}$ by

$$S_1 = \{P \in \tilde{S};\, p(P) \geqq 1,\, q(P) \geqq 1\} \cup \{P \in \tilde{S};\, r(P) \geqq 1,\, s(P) \geqq 1\},$$
$$S_2 = \{P \in \tilde{S};\, p(P) \geqq 1,\, q(P) \leqq 1\} \cup \{P \in \tilde{S};\, r(P) \geqq 1,\, s(P) \leqq 1\},$$
$$S_3 = \{P \in \tilde{S};\, p(P) \leqq 1,\, q(P) \geqq 1\} \cup \{P \in \tilde{S};\, r(P) \leqq 1,\, s(P) \geqq 1\},$$
$$S_4 = \{P \in \tilde{S};\, p(P) \leqq 1,\, q(P) \leqq 1\} \cup \{P \in \tilde{S};\, r(P) \leqq 1,\, s(P) \leqq 1\}.$$

Since they are g-invariant, their images $\bar{S}_k = \pi(S_k)$ are closed subsets of S. It is clear that $\bar{S}_4$ contains $A + B$, hence $\bar{S}_4^0$ is not empty. We shall show that $\bar{S}_k^0$ is empty $(k = 1, 2, 3)$ where $\bar{S}_k^0$ denotes the interior of $\bar{S}_k$.

Let

$$X + iY = (\omega \log Y_0 + \log X_0)/2\pi i,$$
$$U + iV = (\omega' \log Y_0 + \log X_0)/2\pi i$$

where X, Y, *U and V are real. It follows that*

$$X = (\omega \log Y_0 \bar{Y}_0^{-1} + \log X_0 \bar{X}_0^{-1})/4\pi i,$$
$$Y = -(\omega \log Y_0 \bar{Y}_0 + \log X_0 \bar{X}_0)/4\pi,$$
$$U = (\omega' \log Y_0 \bar{Y}_0^{-1} + \log X_0 \bar{X}_0^{-1})/4\pi i,$$
$$V = -(\omega' \log Y_0 \bar{Y}_0 + \log X_0 \bar{X}_0)/4\pi.$$

Y and V are single-valued on $\tilde{S} - C - D$. Define $\Omega = -Y dX \wedge dU \wedge dV$. Then $g^* \Omega = \Omega$ so that it defines a C^{∞} 3-form on $S - A - B$. The form Ω is continuous (well-defined) on a neighborhood of $\bar{S}_k$ $(k = 1, 2, 3)$. If $\bar{S}_k^0$ is not empty, then we have by Stokes' theorem

$$0 < \int_{\bar{S}_k} d\Omega = \int_{\partial \bar{S}_k} \Omega = \int_{\{Y=0\} \cup \{V=0\}} \Omega = 0.$$

In fact, $d\Omega$ is a volume form on $S - A - B$ and $\Omega_{|Y=0} = \Omega_{|V=0} = 0$. Thus $\bar{S}_k^0$ is empty, $S = \bar{S}_4$. Therefore $p(P) \leqq 1$, $q(P) \leqq 1$. Since p and q satisfy the maximum principle, we infer that on $\tilde{S}$

$$p(P) < 1, \quad q(P) < 1.$$

Hence $0 \leqq Y^{-1} < \infty$, $0 \leqq V^{-1} < \infty$. Letting $\Omega' = -Y^{-1} dX \wedge dU \wedge dV^{-1}$, we have $g^* \Omega' = \Omega'$ and $d\Omega' = dX \wedge dY^{-1} \wedge dU \wedge dV^{-1}$ is a volume form on S. Again by Stokes' theorem,

$$0 < \int_S d\Omega' = \int_{\partial S} \Omega' = 0$$

which is a contradiction.

Next we consider the case where $\beta < 0$, $\beta' < 0$. We define

$$\begin{aligned}
S_1 &= \{p \in \tilde{S}; p(P) \geqq 1, q(P) \geqq 1\} \cup \{P \in \tilde{S}; r(P) \leqq 1, s(P) \leqq 1\},\\
S_2 &= \{p \in \tilde{S}; p(P) \geqq 1, q(P) \leqq 1\} \cup \{P \in \tilde{S}; r(P) \leqq 1, s(P) \geqq 1\},\\
S_3 &= \{P \in \tilde{S}; p(P) \leqq 1, q(P) \geqq 1\} \cup \{P \in \tilde{S}; r(P) \geqq 1, s(P) \leqq 1\},\\
S_4 &= \{P \in \tilde{S}; p(P) \leqq 1, q(P) \leqq 1\} \cup \{P \in \tilde{S}; r(P) \geqq 1, s(P) \geqq 1\},\\
\bar{S}_k &= \pi(S_k), \quad \bar{S}_k^0 = \text{the interior of } S_k.
\end{aligned}$$

We define X, Y, U, V and Ω in the same way as before. Then the form Ω is continuous (well-defined) on a neighborhood of $\bar{S}_2$ and $\bar{S}_3$ so by the same argument as before we infer that $\bar{S}_2^0$ and $\bar{S}_3^0$ are empty. Clearly $\bar{S}_1$ and $\bar{S}_4$ contain B and A respectively. Their boundaries $\partial\bar{S}_1$ and $\partial\bar{S}_4$ are the same closed subset $\pi(p^{-1}(1) \cap q^{-1}(1))$, therefore of at most dimension 2. Hence $\partial\bar{S}_1 = \partial\bar{S}_4 = \phi$, that is, $\bar{S}_1$ and $\bar{S}_4$ determine topological 4 cycles with $\bar{S}_1 \cap \bar{S}_4 = \phi$. This is a contradiction. Thus we have $\beta\beta' < 0$, $\det B = -1$. Q.E.D.

(8.13) **Lemma.**

(8.13.1) $\omega = [[\overline{n_0, n_1, \cdots, n_{r-1}}]]$, $\omega^* = [[\overline{m_0, m_1, \cdots, m_{s-1}}]]$.

(8.13.2) *Let* $\omega^{-1} = [[e'_0, \cdots, e'_{t-1}, \overline{m'_0, \cdots, m'_{s'-1}}]]$ *be the modified continued fraction expansion of* ω^{-1}. *Then (we may assume)* $s = s'$, $m_\nu = m'_{\nu+l}$ *for some* l.

Proof. By (8.11), $\omega = c(\alpha - a)^{-1}$ and satisfies the equation

$$b\omega^2 + (d-a)\omega - c = 0.$$

Since $\omega > \omega'$, ω is the larger root of it. Letting $x = [[\overline{n_0, \cdots, n_{r-1}}]]$, then we infer that $x > x'$ (= the conjugate of x), $bx^2 + (d-a)x - c = 0$. Hence $\omega = x$. Similarly we have $\omega^* = [[\overline{m_0, \cdots, m_{s-1}}]]$. By (8.11) and (8.12) we have $\mathbf{Z} + \mathbf{Z}\omega^* = \beta(\mathbf{Z} + \mathbf{Z}\omega)$ and $\beta\beta' < 0$. Notice that s may not be the smallest period. If s' is the smallest period of the expansion of ω^{-1}, then s' divides s. So we may take s instead of s' in (8.13.2). The second assertion follows from [5] Proposition 1.1. (iii). Q.E.D.

(8.14) If $\beta < 0$ and $\beta' > 0$, then we define

$$\begin{aligned}
&\delta^* = (\omega^{*\prime})^{-1}, \quad \delta = (\omega')^{-1}, \quad \eta = ((\omega^*)^{-1}\omega\beta)',\\
&s_\lambda = X_{-\lambda}, \quad t_\lambda = Y_{-\lambda}, \quad u_\nu = Z_{-\nu}, \quad v_\nu = W_{-\nu},\\
&\psi_1 = \delta^* u_0^{-1} du_0 + v_0^{-1} dv_0 = \eta(\delta s_0^{-1} ds_0 + t_0^{-1} dt_0),\\
&\psi_2 = (\delta^*)' u_0^{-1} du_0 + v_0^{-1} dv_0 = \eta'(\delta' s_0^{-1} ds_0 + t_0^{-1} dt_0).
\end{aligned}$$

We define $C'_\lambda = C_{-\lambda}$, $D'_\nu = D_{-\nu}$ and an automorphism g' of $\tilde{S}$ by $g' = g^{-1}$. Then we have $g'(C'_\lambda) = C'_{\lambda-r}$, $g'(D'_\nu) = D'_{\nu-s}$, $g'^*\psi_1 = \alpha\psi_1$, $g'^*\psi_2 = \alpha^{-1}\psi_2$. Since $\eta > 0$, $\eta' < 0$, this is reduced to the case where $\beta > 0$, $\beta' < 0$. So we may assume that $\beta > 0$, $\beta' < 0$. By (8.13.2) we may assume that $m_\nu = m'_\nu$ for any ν by replacing

D_ν by $D_{\nu-l}$. Let $U^+(\omega)$ be the group of totally positive units keeping $\mathbf{Z}+\mathbf{Z}\omega$ invariant. Let α_0 be the generator of $U^+(\omega)$ with $\alpha_0 > 1$, and $\beta_0 = (\eta_1 \cdots \eta_t)/\omega$. (See [5] p. 93.) Define the integral matrix B_0 by

$$(\omega^*, 1)\, B_0 = \beta_0 (\omega, 1)\,.$$

By [5] p. 93, $\beta_0 > 0$, $\beta_0' < 0$, $\det B_0 = -1$. By (8.11) and our assumption $\beta\beta_0^{-1} \in U^+(\omega)$. Consequently $\beta = \beta_0 \alpha_0^l$ for some l. Let r_0 and s_0 be the smallest period of modified continued fraction expansion of ω and ω^* respectively. By replacing D_ν by $D_{\nu - l s_0}$, we may assume that $\beta = \beta_0$. Since α belongs to $U^+(\omega)$ by (8.11), we have $\alpha = \alpha_0^n$, $r = n r_0$, $s = n s_0$ for some n.

(8.15) Now we are in a position to prove (8.1) Theorem. With the notations in §1, we define a mapping $\tilde f$ of $\tilde S$ into $\mathscr{U}$ by

$$(x_\lambda, y_\lambda, z_\nu, w_\nu) = (X_\lambda, Y_\lambda, Z_\nu, W_\nu)\,.$$

Evidently $f \circ g = g_0^n \circ f$. Let

$$\begin{aligned}
S_1 &= \{P \in \tilde S;\, p(P) \geqq 1,\, q(P) \geqq 1\} \cup \{P \in \tilde S;\, r(P) \geqq 1,\, s(P) \leqq 1\}\,,\\
S_2 &= \{P \in \tilde S;\, p(P) \geqq 1,\, q(P) \leqq 1\} \cup \{P \in \tilde S;\, r(P) \geqq 1,\, s(P) \geqq 1\}\,,\\
S_3 &= \{P \in \tilde S;\, p(P) \leqq 1,\, q(P) \geqq 1\} \cup \{P \in \tilde S;\, r(P) \leqq 1,\, s(P) \leqq 1\}\,,\\
S_4 &= \{P \in \tilde S;\, p(P) \leqq 1,\, q(P) \leqq 1\} \cup \{P \in \tilde S;\, r(P) \leqq 1,\, s(P) \geqq 1\}\,,\\
\bar S_k &= \pi(S_k), \qquad \bar S_k^0 = \text{the interior of } \bar S_k\,.
\end{aligned}$$

Define X, Y, U, V and Ω as in (8.12). By the same argument as before we infer that $\bar S_1^0 = \bar S_2^0 = \phi$. Clearly $\bar S_3$ and $\bar S_4$ contain B and A respectively. Therefore $p(P) \leqq 1$ and p is continuous on $\tilde S$ by continuating by $p = r^{1/\beta}$ on $\tilde S - C$. By the maximum principle we have $p(P) < 1$. Consequently $\tilde f$ is a holomorphic mapping of $\tilde S$ into $\mathscr{D}$ with $\tilde f \circ g = g_0^n \circ \tilde f$. Therefore $\tilde f$ induces a holomorphic mapping f of S into $S_\omega^{[n]}$. Since f is everywhere of maximal rank and proper, f is surjective. Since $\chi(S) = r + s = n(r_0 + s_0) = \chi(S_\omega^{[n]})$ by (6.1) and (8.14), f is of degree one, that is, an isomorphism of S onto $S_\omega^{[n]}$. Q.E.D.

§9. Half Inoue surfaces

(9.1) **Theorem.** *Let S be a* VII_0 *surface with a cycle C of rational curves. Suppose that $C^2 < 0$, and there exists a nontrivial flat line bundle L on S with L_C trivial. Then S is isomorphic to a half Inoue surface $\hat S_\omega^{[2n+1]}$.*

Proof. We keep the notations in §8. By (2.13.1) L is contained in $\operatorname{Ker}(H^1(S, \mathbf{C}^*) \to H^1(C, \mathbf{C}^*)) \cong \operatorname{Ker}(H^1(S, \mathbf{C}^*) \to H^1(C, \mathcal{O}_C^*))$ and therefore it is of order two. Let $S^* = \operatorname{Specan}(\mathcal{O}_S \oplus L)$, $f\colon S^* \to S$ be the natural projection. Then $f^{-1}(C)$ is a disjoint union of two copies of C. By (8.1) S^* is a hyperbolic Inoue surface, $b_2(S^*) = \#(\text{irreducible components of } f^{-1}(C))$. Hence $b_2(S) = \#(\text{irreducible components of } C)$. By (3.12) we have flat line bundles F and L' such that L'_C is trivial, $N(S) = \{F, -F + L'\}$. Since $\operatorname{Ker}(H^1(S, \mathbf{C}^*) \to H^1(C, \mathbf{C}^*))$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$, we have $L = L'$. Let σ be a covering transformation of f, $\mathscr{D}$ the universal covering of

S^*, τ a lifting of σ to $\mathscr{D}$. Let φ_1 and φ_2 be nontrivial elements of $H^0(S, \Omega_S^1(\log C)(F))$ and $H^0(S, \Omega_S^1(\log C)(-F+L))$ respectively, $\omega_\nu = f^* \varphi_\nu$, $\tilde{\omega}_\nu$ the pull-back of ω_ν to $\mathscr{D}$. Then $\varphi_1 \wedge \varphi_2$ is contained in $H^0(S, \Omega_S^2(L))$ so that $\sigma^*(\omega_1 \wedge \omega_2) = -\omega_1 \wedge \omega_2$. Since $\sigma^* f^* F^* = f^* F$, we have $\sigma^* \omega_1 = \delta_1 \omega_1$, $\sigma^* \omega_2 = \delta_2 \omega_2$ for some constants δ_1 and δ_2. Since an infinite chain of rational curves on $\mathscr{D}$ is transformed onto another by τ, we may assume $\tau(V_0) \subset W_0$ (see §1). Moreover the coordinates x_0, y_0, z_0 and w_0 are uniquely (up to constant multiples) determined by the expressions of the form (8.11.1) in view of (8.3). We have therefore either 1) $\tau^* w_0 = c_1 y_0$, $\tau^* z_0 = c_2 x_0$ or 2) $\tau^* w_0 = c_1 x_0$, $\tau^* z_0 = c_2 y_0$. In the case 2),

$$\begin{aligned} \tilde{\omega}_1 \wedge \tilde{\omega}_2 &= (\omega^* - \omega^{*\prime})\, w_0^{-1} z_0^{-1}\, dw_0 \wedge dz_0 \\ &= -(\omega - \omega')\, y_0^{-1} x_0^{-1}\, dy_0 \wedge dx_0, \\ \tau^*(\tilde{\omega}_1 \wedge \tilde{\omega}_2) &= -(\omega^* - \omega^{*\prime})\, y_0^{-1} x_0^{-1}\, dy_0 \wedge dx_0. \end{aligned}$$

Since $\omega^* > \omega^{*\prime}$, $\omega > \omega'$, we have $\sigma^*(\omega_1 \wedge \omega_2) = \omega_1 \wedge \omega_2$, which is a contradiction. So the case 1) occurs. In view of (8.11.1) $\tau^* \tilde{\omega}_1 = \omega^* y_0^{-1} dy_0 + x_0^{-1} dx_0 = \delta_1 \beta(\omega y_0^{-1} dy_0 + x_0^{-1} dx_0)$, hence we have $\omega = \omega^*$, $\delta_1 \beta = 1$. Similarly $\delta_2 \beta' = 1$, therefore $\delta_2 = \delta_1'$. By (8.8) $w_0 = y_0^e x_0^f$, $z_0 = y_0^g x_0^h$. Therefore $\tau^* y_0 = y_0^{-h} x_0^f$, $\tau^* x_0 = y_0^g x_0^{-e}$ by modifying x_0 and y_0 by constant multiples. Hence we have $(\omega, 1) B = \beta(\omega, 1)$. Since $\beta\beta' = \delta_1 \delta_2 = -1$, β is an element of $U(\omega) - U^+(\omega)$. There exists an integer m such that $\beta = \beta_0^{2m+1}$ for a generator β_0 of $U(\omega)$ and $\tau = \tau_0^{2m+1}$ (see (1.6)). On the other hand $\pi_1(S^*)$ is generated by g, a transformation of $\mathscr{D}$. The transformation g is a multiple of g_0, say, $g = g_0^l = \tau_0^{2l}$ for some l. Thus the surface S is a quotient of $\mathscr{D}$ by a group G generated by τ_0^{2l} and τ_0^{2m+1}. However τ^2 is contained in $\{\tau_0^{2lk};\, k \in \mathbf{Z}\}$ which shows that l divides $2m+1$. Let $2m+1 = lk$. Then both l and k are odd integers so that G is generated by $\tau_0^l (= \tau_0^{2n+1})$, S is isomorphic to $\hat{S}_\omega^{[2n+1]} = \mathscr{D}/\{\tau_0^{(2n+1)k'};\, k' \in \mathbf{Z}\}$. Q.E.D.

(9.2) **Theorem.** *Let S be a* VII_0 *surface with a cycle C of rational curves with $C^2 < 0$. Suppose that S satisfies one of the following conditions.*

(9.2.1) *There exists a flat line bundle L such that $K_S + C = L$.*

(9.2.2) $b_2 = \#$ *(irreducible components of C).*

(9.2.3) $C^2 \leqq -b_2$.

(9.2.4) $[H_1(S, \mathbf{Z}) : i_* H_1(C, \mathbf{Z})] \neq 1$.

Then S is isomorphic to a half Inoue surface.

Proof. Assume (9.2.1). Then L is a nontrivial flat line bundle with L_C trivial. Indeed, $L_C = (K_S + C)_C \cong \mathcal{O}_C$, and if L is trivial, then $h^1(C, \mathcal{O}_C) = 2$ follows from the proof of (2.2.1) and (2.6), which is a contradiction. So L is nontrivial. Hence S is a half Inoue surface by (9.1). Assume (9.2.2). Then irreducible components C_λ of C form a $\mathbf{Q}$-basis of $H^2(S, \mathbf{Q})$. Since $(K_S + C)\, C_\lambda = 0$ for any irreducible component C_λ of C, the line bundle $K_S + C$ is flat, namely, we have (9.2.1). Hence S is a half Inoue surface. Assume (9.2.3). Compute $(K_S + C)^2$. We have $0 \geqq (K_S + C)^2 = (K_S + C) K_S = -b_2 - C^2$. Hence $C^2 \geqq -b_2$. By the assumption $C^2 = -b_2$, and $K_S + C$ is flat by (2.5). Hence S is a half Inoue surface. Assume

(9.2.4). From (2.13.1) it follows that there exists a nontrivial flat line bundle L with L_C trivial. Hence S is a half Inoue surface by (9.1). Q.E.D.

(9.3) **Theorem.** *Let S be a* VII_0 *surface with a cycle C of rational curves. Then $C^2 \geqq -b_2$. Equality holds if and only if S is isomorphic to a half Inoue surface.*

This follows from the proof of (9.2).

Notice that $C^2 \leqq 0$ and equality holds if and only if S is an "exceptional" compactification of an affine bundle over an elliptic curve (see [2]).

§10. Surfaces with elliptic curves

(10.1) **Theorem.** *Let S be a* VII_0 *surface with no meromorphic functions except constants. Suppose that S has an elliptic curve but no cycles of rational curves. Then S is isomorphic to a Hopf surface.*

Proof. Assume first that S has two elliptic curves. Then by (5.2) S is isomorphic to a Hopf surface. Next consider the case where there exists only one elliptic curve E on S. Let F be an element of infinite order in $\mathrm{Ker}\,(H^1(S, \mathbf{C}^*) \to H^1(E, \mathcal{O}_E^*))$. We may assume $H^0(S, F) = 0$ by taking $-F$ instead of F if $H^0(S, F) \neq 0$. Consider the exact sequence,

$$\begin{aligned}0 &\to H^0(S, F-[E]) \to H^0(S, F) \to H^0(E, \mathcal{O}_E)\\ &\to H^1(S, F-[E]) \to H^1(S, F) \to H^1(E, \mathcal{O}_E)\\ &\to H^2(S, F-[E]) \to H^2(S, F) \to 0.\end{aligned}$$

It follows $H^0(S, F-[E]) = 0$, $\chi(S, F-[E]) = 0$. Therefore we have

$$h^0(K_S+[E]-F) = h^2(S, F-[E]) = h^1(S, F-[E]) \geqq h^0(E, \mathcal{O}_E) = 1.$$

Hence there exists an effective divisor $C = \sum n_i C_i (C_i \neq E)$ and an integer $r (r \geqq 0)$ such that

$$K_S + [E] - F = [C + rE].$$

Since $K_S C_i \geqq 0$, $C_i^2 \leqq 0$, $(K_S+E)E = 0$, $EC \geqq 0$, we have

$$(C+rE)^2 = (K_S+E-F)(C+rE) = (K_S+E)C \geqq 0.$$

But since $L^2 \leqq 0$ for any line bundle L on S, we have $(C+rE)^2 = 0$, so that $[C+rE]$ is flat in view of (2.5). If $r > 0$, then $[E]$ is flat by (2.10) so that $[C]$ is flat. Therefore if $C \neq 0$, then C_{red} is either a smooth elliptic curve or a cycle of rational curves, which contradicts our assumptions. Hence $C = 0$, $K_S + [E] - F = [rE]$. This shows $c_1^2 = 0$. If $r = 0$, then C itself is flat. If $C \neq 0$, then we have a contradiction again by the same argument. Hence $K_S + [E] - F = 0$. We shall show $E^2 = 0$. Assume the contrary. Then, by (2.10) we have $H^0(S, F') = 0$ for any flat line bundle F' on S. Consider the exact sequence;

$$\begin{aligned}0 &\to H^0(S, -K_S-2[E]) \to H^0(S, -K_S-[E]) \to H^0(E, \mathcal{O}_E)\\ &\to H^1(S, -K_S-2[E]) \to H^1(S, -K_S-[E]) \to H^1(E, \mathcal{O}_E)\\ &\to H^2(S, -K_S-2[E]) \to H^2(S, -K_S-[E]) \to 0.\end{aligned}$$

Since $H^0(S, 2F) = H^0(S, -F) = 0$, we have $H^0(S, -K_S - [E]) = 0$, hence $H^0(S, -K_S - 2[E]) = H^2(S, -K_S - 2[E]) = 0$. Since $\chi(S, -K_S - 2[E]) = \chi(S, -K_S - [E]) - \chi(E, \mathcal{O}_E) = \chi(S, F) - \chi(E, \mathcal{O}_E) = 0$, we have $H^1(S, -K_S - 2[E]) = 0$. However $h^0(E, \mathcal{O}_E) = 1$ which is absurd. Therefore $E^2 = 0$ so that $c_1^2 = K_S^2 = ([E] - F)^2 = 0$. Thus in any case we have $b_2 = c_1^2 = 0$. Hence by [12, II, Theorem 34], S is isomorphic to a Hopf surface. Q.E.D.

(10.2) **Theorem.** *Any* VII_0 *surface with an elliptic curve is isomorphic to one of* VII_0 *elliptic surfaces, Hopf surfaces and parabolic Inoue surfaces.*

Proof. Assume that S has no meromorphic functions except constants. If there is no cycle of rational curves on S, then S is isomorphic to a Hopf surface by (10.1). If there is a cycle of rational curves, then S is isomorphic to a parabolic Inoue surface by (7.1). Q.E.D.

(10.3) **Table (of surfaces of class VII_0 with curves)**

Curves	Surfaces
1) (more than) 3 elliptic curves	elliptic VII_0 surfaces
2) two elliptic curves	Hopf surfaces
3) an elliptic curve and no cycles	Hopf surfaces
4) an elliptic curve and a cycle	parabolic Inoue surfaces
5) two cycles	hyperbolic Inoue surfaces
6) a cycle C with $C^2 = 0$ and no elliptic curves	exceptional compactifications with no elliptic curves
7) a cycle C with $C^2 < 0$	
7.1) $b_2(S) = b_2(C)$	half Inoue surfaces
7.2) $b_2(S) > b_2(C)$	(examples exist)
8) no elliptic curves, no cycles	?

The above table is made by combining the results of this article and [2, 12]. See (5.2), (7.1), (8.1), (9.2), (10.1). 1) and 2) are due to Kodaira [12] and Kato respectively. 4) is due to Enoki [2] and the author independently. 6) is due to [2]. 2), 3), 4), 5) and 7.1) are proved in this article. See [19] for the definition of exceptional compactifications.

It is still unknown whether any VII_0 surface with curves has a cycle or an elliptic curve.

§11. Surfaces with $b_2 = 1$

(11.1) **Lemma.** *Let S be a* VII_0 *surface with $b_2 = 1$. Suppose that S has a curve C. Then C is an elliptic curve or a rational curve with a node.*

Proof. Suppose not. Hence C is a nonsingular rational curve in view of (2.2). Since $c_1^2 = -b_2 = -1$, K_S is a $\mathbf{Q}$-basis of $H^2(S, \mathbf{Q})$. So we write $C = mK_S$. It follows that $m = 1$ or -2. If $m = 1$, then C is an exceptional curve of the first kind which contradicts the assumption. So $m = -2$. There exists a flat line bundle F such that

$C = -2K_S + F$. Since $H^1(S, \mathbf{C}^*) \cong \mathbf{C}^* +$ (torsions), we have flat line bundles G and F' such that $F = 2G + F'$, F' is of order l. *Let* $S' := \operatorname{Specan}\left(\bigoplus_{k=0}^{l-1} F'^k\right)$, $f: S' \to S$ be the canonical projection. Then $f^*C = -2K_{S'} + 2f^*G$ because $f^*F' = 0$. The curve f^*C is a disjoint union of l copies of C because F'_C is trivial. Let $S^* = \{(x, \zeta) \in -K_{S'} + f^*G\colon\ g(x) = \zeta^2\}$ where $g(\neq 0) \in H^0(S', -2K_{S'} + 2f^*G)$, $(g) = f^*C$. Then S^* is a VII_0 surface with $b_2(S^*) = 0$. In fact $b_2(S^*) = \chi(S^*) = 2\chi(S') - \chi(f^*C) = l(2\chi(S) - \chi(C)) = 0$. However the inverse image of f^*C has negative selfintersection number, which contradicts $b_2(S^*) = 0$. Q.E.D.

(11.2) **Theorem.** *Any* VII_0 *surface with* $b_2 = 1$ *and at least one curve is isomorphic to one of the following;*

half Inoue surface $\hat{S}^{[1]}_{\omega}$, $\omega = (3 + \sqrt{5})/2$.
compactifications of affine bundles of degree one over nonsingular elliptic curves by rational curves with nodes.

Proof. Let S be a VII_0 surface with $b_2 = 1$, and at least one curve. If S contains an elliptic curve and no rational curves, then S is a Hopf surface in view of (10.1) which contradicts $b_2 = 1$. Hence S must contain a rational curve C. Then it must be a rational curve with a node in view of (11.1). If $C^2 = 0$, then S is a compactification of an affine bundle of degree one over an elliptic curve by C by [2, Main Theorem]. If $C^2 < 0$, then S is isomorphic to a half Inoue surface $\hat{S}^{[n]}_{\omega}$ for certain ω and n. Since $C^2 = c_1^2 = -b_2 = -1$ in view of (9.3), the continued fraction expansion associated to C is $[[\bar{3}]]$, hence $n = 1$, $\omega = (3 + \sqrt{5})/2$. Q.E.D.

§12. Surfaces with antipluricanonical divisors

(12.1) **Lemma.** *Let S be a* VII_0 *surface with* $b_2 > 0$. *Then plurigenera* $P_m = 0$.

Proof. Suppose not. Then there exists an effective divisor D and a positive integer m such that $D = mK_S$. For any irreducible curve E we have $K_S E \geqq 0$. Hence $D^2 = mK_S D \geqq 0$. By (2.1.2) $D^2 = 0$, $c_1^2 = K_S^2 = -b_2 = 0$. Q.E.D.

(12.2) **Lemma.** *Let S be a* VII_0 *surface with an elliptic curve. (S may have nonconstant meromorphic functions.) Suppose that* $P_m = 0$ *for any positive integer m, any unramified Galois covering S' of S is cyclic and* $H_1(S', \mathbf{Z}) \cong \mathbf{Z}$. *Then S is a primary Hopf surface or a parabolic Inoue surface.*

Proof. If S has no meromorphic functions except constants, then S is a primary Hopf surface or a parabolic Inoue surface by (7.1) and (5.2). If S has nonconstant meromorphic functions then S is an elliptic VII_0 surface, hence in particular it has two elliptic curves. Therefore S is a primary Hopf surface by (5.3). Q.E.D.

(12.3) **Theorem.** *Let S be a* VII_0 *surface with a cycle C of rational curves. Suppose* $H^2(S, \Theta_S(-\log C)) = 0$ *and that there exists an irreducible divisor E with* $EC > 0$. *Then there exists a smooth proper family* $\pi: \mathscr{S} \to D$ *over the unit disc with a divisor* $\mathscr{C}$ *flat over D such that* $\mathscr{S}_0 = S$, $\mathscr{C}_0 = C$, $\mathscr{S}_t (t \neq 0)$ *is a blown-up primary Hopf surface or a blown-up parabolic Inoue surface,* $\mathscr{C}_t (t \neq 0)$ *is a nonsingular elliptic curve.*

Proof. Consider the exact sequence (cf. (4.2).)

$$0 \to \Theta_S(-\log C) \to \Theta_S \to J_C \to 0.$$

We have an exact sequence,

$$\begin{aligned} &\to H^1(S, \Theta_S(-\log C)) \to H^1(S, \Theta_S) \to H^1(C, J_C) \\ &\to H^2(S, \Theta_S(-\log C)) \to H^2(S, \Theta_S) \to 0. \end{aligned}$$

By the assumption we have $H^2(S, \Theta_S) = 0$ and $H^1(S, \Theta_S) \to H^1(C, J_C)$ is surjective. This implies that the canonical restriction homomorphism $H^1(S, \Theta_S) \to H^1(U, \Theta_U)$ for a strictly pseudoconvex open neighborhood U of C is surjective by (4.3). This implies that there exists a smooth proper family $\pi: \mathcal{S} \to D$ over the unit disc with a π-flat divisor $\mathcal{C}$ such that $\mathcal{S}_0 = S$, $\mathcal{C}_0 = C$, $\mathcal{C}_t$ $(t \neq 0)$ is a nonsingular elliptic curve. Since $P_m(S) = 0$ for any positive integer m by (12.1), we have $P_m(\mathcal{S}_t) = 0$ for any t. Since there exists an irreducible divisor E such that $EC > 0$, we have $H_1(S, \mathbf{Z}) = i_*(H_1(C, \mathbf{Z}))$ and $H_1(S', \mathbf{Z}) = (i')_* H_1(C', \mathbf{Z})$ for any unramified Galois covering $\pi: S' \to S$ and $C' = \pi^{-1}(C)$ in view of (2.14). Since the covering group of $\pi_{C'}$ coincides with that of π, π must be cyclic. Since $\mathcal{S}_t$ is diffeomorphic to S, all the assumptions of (12.2) except minimality are satisfied so that $\mathcal{S}_t$ is either a blown-up primary Hopf surface or a blown-up parabolic Inoue surface. Q.E.D.

(12.4) **Lemma.** *Let S be a* VII$_0$ *surface with $b_2 > 0$. Suppose there is a divisor D such that $mK_S + D = 0$ for a certain positive integer m. Then D_{red} contains a cycle C of rational curves and $H^2(S, \Theta_S(-\log C)) = 0$.*

Proof. It is easy to see that D is an effective divisor. First we assume $m = 1$. Then $h^1(D, \mathcal{O}_D) = 2$ by (2.6) and D_{red} contains an elliptic curve or a cycle of rational curves in view of (2.7). If S has no cycles of rational curves, then S is an elliptic surface or a Hopf surface, so $b_2 = 0$ which contradicts the assumption. Hence S has a cycle C of rational curves. If S has an elliptic curve E, then S is a parabolic Inoue surface and $K_S = -C - E$, hence $D = C + E$ contains C. If S has two cycles A and B of rational curves then S is a hyperbolic Inoue surface and $K_S = -A - B$, C is either A or B. Therefore in this case $D = A + B$ contains C too. If S has only one cycle of rational curves, then D contains C because $h^1(D_{\text{red}}, \mathcal{O}_{D_{\text{red}}}) \geqq 1$ and D_{red} cannot be a tree by (2.3). Moreover by the Serre duality, in these cases we have $h^2(S, \Theta_S(-\log C)) = h^0(S, \Omega^1_S(\log C) \otimes \Omega^2_S) \leqq h^0(S, \Omega^1_S(C - D)) \leqq h^0(S, \Omega^1_S) = 0$, hence $H^2(S, \Theta_S(-\log C)) = 0$. Assume next $m \geqq 2$. Then we consider an analytic subvariety S' of the anticanonical line bundle $-K_S$ defined by $S' = \{(x, \zeta) \in -K_S; \zeta^m = f(x)\}$ where f is a defining equation of D. Then there is a meromorphic 2-form dx/ζ on $-K_S$. Let S^* be the minimal resolution of S', $\pi; S^* \to S$ the canonical projection of S^* onto S. Then $\pi^*(dx/\zeta)$ is a nontrivial meromorphic 2-form on S^* so that there exists an effective divisor D^* of S^* such that $\pi(D^*) = D$, $K_{S^*} + D^* = 0$. We shall show S^* is of class VII. Since $b_2 > 0$, S has no meromorphic functions except constants, hence nor has S^*. Therefore by the classification table S^* is a blown-up $K3$ surface or a blown-up complex torus or a blown-up surface of class VII. However in the first two cases K_{S^*} is always effective, hence $D^* = 0$, $D = 0$ which is a contradiction. Hence S^* is of class VII. In view of the assertion in the case where $m = 1$, D^* contains a cycle C^* of rational curves and $\pi(C^*)$ is therefore a cycle of rational curves, hence D contains a cycle of rational curves. Indeed, if

$C := \pi(C^*)$ is not a cycle of rational curves, then C is a tree of rational curves, the support of C is simply connected. Let S_1 be a double covering of S, C_1, D_1 the inverse images of C, D. Then $mK_{S_1} + D_1 = 0$. By the same construction as above we have an m-fold covering S_1' of S_1, and the minimal resolution S_1^* of S_1', an effective divisor D_1^* of S_1^* with $K_{S_1^*} + D_1^* = 0$. By contruction, C_1, D_1, D_1^* consist of two copies of C, D, D^* so $h^1(D_1^*, \mathcal{O}_{D_1^*}) \geqq 4$ which is a contradiction of (2.2.1). Moreover $H^2(S, \Theta_S(-\log C)) = 0$. Because $\pi^* H^0(S, \Omega_S^1(\log C) \otimes \Omega_S^2)$ is contained in $H^0(S^*, \Omega_{S^*}^1(\log \pi^* C) \otimes \Omega_{S^*}^2)$, and $h^0(S^*, \Omega_{S^*}^1(\log \pi^* C) \otimes \Omega_{S^*}^2) = h^0(S^*, \Omega_{S^*}^1(\log(\pi^* C)_{\mathrm{red}}) \otimes \Omega_{S^*}^2) \leqq h^0(S^*, \Omega_{S^*}^1) = 0$, hence $H^0(S, \Omega_S^1(\log C) \otimes \Omega_S^2) = 0$. So $H^2(S, \Theta_S(-\log C)) = 0$. Q.E.D.

(12.5) **Theorem.** *Let S be a* VII_0 *surface with $b_2 > 0$. Suppose S has an antipluricanonical divisor D, that is, $mK_S + D = 0$ for a certain positive integer m. Then there exists a proper flat family $\pi \colon \mathcal{S} \to D$ over the unit disc D with a π-flat divisor $\mathcal{C}$ such that*

(12.5.1) *$\mathcal{S}_0 = S$, $\mathcal{S}_t (t \neq 0)$ is a blown-up primary Hopf surface,*

(12.5.2) *$\mathcal{C}_0$ is a cycle of rational curves, $\mathcal{C}_t (t \neq 0)$ is a nonsingular elliptic curve.*

Proof. If S has two cycles of rational curves, then the assertions follow from (5.4), (5.10). So we may assume that S has only one cycle of rational curves. By (12.4) S has a unique cycle contained in D_{red}. If moreover there is an irreducible curve E with $EC > 0$, then a desired family of deformations exists by (12.3), (12.4), (5.11) and the openness of versality of deformations. So we consider the case where there is no irreducible curve E with $EC > 0$. Suppose moreover $C^2 = 0$. Since there is an m-fold ramified covering S^* of S with $K_{S^*} + D^* = 0$. Let $D = C' + D'$ for effective divisors $C' (\subset C)$ and D' with $D' \cap C = \phi$. Then $(C')^2 = (-mK_S - D')C' = mCC' = 0$ because $[C]$ is flat and $[K_S + C]_C = \mathcal{O}_C$. Hence $C' = kC$ for some k by (2.10) and (2.12). It is easy to see that $[C]_C$ is of infinite order. (See (7.4).) Since $[-mC + C']_C = [mK_S + C']_C = [D']_C = \mathcal{O}_C$, we have $m = k$. Hence by the construction of S^* in (12.4), D^* contains a cycle C^* with $(C^*)^2 = 0$. Therefore S^* has an elliptic curve E^* such that $D^* = C^* + E^*$, so S^* is a blown-up parabolic Inoue surface by (7.1). Then the image of E^* in S is an elliptic curve by a similar proof to (12.4). Hence S itself is a parabolic Inoue surface, hence $m = 1$, $S^* = S$ and a desired family of deformations of S exists by (5.11). Finally we suppose $C^2 < 0$ and there is no irreducible curve E with $EC > 0$. If moreover $H_1(S, \mathbf{Z}) \neq i_* H_1(C, \mathbf{Z})$ for the inclusion i of C into S, then S is a half Inoue surface by (9.2), hence $2K_S + 2C = 0$, $\pi_1(S) \cong \mathbf{Z}$. Therefore any unramified Galois covering $f; S^* \to S$ is cyclic and $H_1(S^*, \mathbf{Z}) \cong \mathbf{Z}$. Although one of the assumptions in (12.3) is not satisfied, the proof there is applied. Hence a desired family of deformations of S exists. If $H_1(S, \mathbf{Z}) \cong i_* H_1(C, \mathbf{Z})$, then for any unramified covering $f; S^* \to S$, we have $H_1(S^*, \mathbf{Z}) \cong i_* H_1(f^* C, \mathbf{Z}) \cong \mathbf{Z}$ and S^* is a VII_0 surface with $mK_{S^*} + D^* = 0$. Therefore the proof of (12.3) is applied to show that a desired family of deformations of S exists. Q.E.D.

(12.6) **Corollary.** *Let S be a* VII_0 *surface with $b_2 > 0$ and an antipluricanonical effective divisor. Then S is diffeomorphic to a blown-up primary Hopf surface.*

References

1. Artin, M.: Some numerical criteria for contractibility of curves on algebraic surfaces. Amer. J. Math. **84**, 485–496 (1964)
2. Enoki, I.: Surfaces of class VII_0 with curves. Tohoku Math. J. **33**, 453–492 (1981)
3. Grauert, H.: Über Modifikationen und exzeptionelle analytische Mengen. Math. Ann. **146**, 331–368 (1962)
4. Inoue, M.: On surfaces of class VII_0. Invent. Math. **24**, 269–310 (1974)
5. Inoue, M.: New surfaces with no meromorphic functions. Proc. Int. Cong. Math., Vancouver, vol. 1, 423–426 (1974)
6. Inoue, M.: New surfaces with no meromorphic functions, II. Complex Analysis and Algebraic Geometry, Iwanami Shoten Publ. and Cambridge Univ. Press, pp. 91–106 (1977)
7. Karras, U.: Deformations of cusp singularities. Proc. of Symposia in Pure Math. **30**, 37–44 (1977)
8. Kato, Ma.: Compact complex manifolds containing "global spherical shells". I. Proc. Int. Symp. Algebraic Geometry, Kyoto, pp. 45–84 (1977)
9. Kato, Ma.: On a certain class of non-algebraic non-Kähler compact complex manifolds. Preprint
10. Kawamata, Y.: On deformations of compactifiable complex manifolds. Math. Ann. **235**, 247–265 (1978)
11. Kodaira, K.: On compact complex analytic surfaces, I, II. Ann. Math. **71**, 111–152 (1960); **77**, 563–626 (1962)
12. Kodaira, K.: On the structure of compact complex analytic surfaces, I, II, III, IV. Amer. J. Math., **86**, 751–798 (1964); **88**, 682–721 (1966); **90**, 55–83 (1968); **90**, 170–192 (1968)
13. Kodaira, K.: On stability of compact submanifolds of complex manifolds. Amer. J. Math. **85**, 79–94 (1963)
14. Laufer, H.: Normal two dimensional singularities. Annals of Math. Studies, Princeton Univ. Press, (1971)
15. Laufer, H.: Taut two dimensional singularities. Math. Ann. **205**, 131–164 (1973)
16. Laufer, H.: Versal deformation for two dimensional pseudoconvex manifolds. Annali d. Scuola Nurm. Sup. di Pisa **7**, 511–521 (1980)
17. Nakamura, I.: Inoue-Hirzebruch surfaces and a duality of hyperbolic unimodular singularities, I. Math. Ann. **252**, 221–235 (1980)
18. Nakamura, I.: On surfaces of class VII_0 with curves. Proc. Japan. Acad. **58A**, 380–383 (1982)
19. Nakamura, I.: On surfaces of class VII_0 with global spherical shells. Proc. Japan Acad. **59A**, 29–32 (1983)
20. Nakamura, I.: VII_0 surfaces and a duality of cusp singularities, *Classification of algebraic and analytic manifolds*, Progress in Mathematics, vol. 39, pp. 333–378. Birkhäuser 1983
21. Oda, T.: Torus embeddings and applications. Tata Inst. Lecture Notes, Bombay (1978)

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