Teoria Ergódica Diferenciável

lecture 18: Ergodic decomposition theorem

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Radon-Nikodym theorem (reminder)

DEFINITION: Let S be a space equipped with a σ-algebra, and μ, ν two measures on this σ-algebra. We say that ν is **absolutely continuous** with respect to μ if for each measurable set A, $\mu(A) = 0$ implies $\nu(A) = 0$. This relation is denoted $\nu \ll \mu$; clearly, it defines a partial order on measures.

THEOREM: (Radon-Nikodym) Let μ, ν be two measures on a space S with a σ-algebra, satisfying $\mu(S) < \infty$, $\nu(S) < \infty$ and $\nu \ll \mu$. **Then there exists an integrable function $f: S \longrightarrow \mathbb{R}^{\geqslant 0}$ such that $\nu = f\mu$.**

COROLLARY: Let μ, ν be two ergodic measures on (M, Γ) which are not proportional. **Then $\nu \not\ll \mu$ and $\mu \not\ll \nu$.**

Proof: Indeed, otherwise we would have $\nu = f\mu$ or $\mu = f\nu$, where f is a Γ-invariant measurable function. Then f is constant a. e. by ergodicity. ■

Convex cones and extremal rays (reminder)

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $K \subset V$ a subset. We say that K is **convex** if for all $x, y \in K$, the interval $\alpha x + (1-\alpha)y$, $\alpha \in [0,1]$ lies in K. We say that K is **a convex cone** if it is convex and for all $\lambda > 0$, the homothety map $x \longrightarrow \lambda x$ preserves K.

EXAMPLE: Let M be a space equipped with a σ-algebra $\mathfrak{A} \subset 2^M$, and V the space formally generated by all $X \in \mathfrak{A}$. Denote by $\mathcal{S}$ subspace in V^* generated by all finite measures. This space is called **the space of finite signed measures**. **The measures constitute a convex cone in $\mathcal{S}$.**

DEFINITION: **Extreme point** of a convex set K is a point $x \in K$ such that for any $a, b \in K$ and any $t \in [0,1]$, $ta + (1-t)b = x$ implies $a = b = x$. **Extremal ray** of a convex cone K is a non-zero vector x such that for any $a, b \in K$ and $t_1, t_2 > 0$, a decomposition $x = t_1 a + t_2 b$ implies that a, b are proportional to x.

DEFINITION: **Convex hull** of a set $X \subset V$ is the smallest convex set containing X.

EXAMPLE: Let V be a vector space, and $x_1, ..., x_n, ...$ linearly independent vectors. **Simplex** is the convex hull of $\{x_i\}$. Its extremal points are $\{x_i\}$ **(prove it)**.

Ergodic measures as extremal rays (reminder)

Lemma 1: Let (M,μ) be a measured space, and Γ a group which acts ergodically on M. Consider a measure ν on M which is Γ-invariant and satisfies $\nu \ll \mu$. **Then $\nu = \mathrm{const}\cdot\mu$.**

Proof: Radon-Nikodym gives $\nu = f\mu$. The function $f = \frac{\nu}{\mu}$ is Γ-invariant, because both ν and μ are Γ-invariant. Then $f = \mathrm{const}$ almost everywhere. ■

Lemma 2: Let μ_1, μ_2 be measures, $t_1, t_2 \in \mathbb{R}^{>0}$, and $\mu := t_1\mu_1 + t_2\mu_2$. **Then $\mu_1 \ll \mu$.**

Proof: $\mu_1(U) \leqslant t_1^{-1}\mu(U)$, hence $\mu_1(U) = 0$ whenever $\mu(U) = 0$. ■

Ergodic measures as extremal rays 2 (reminder)

THEOREM: Let (M,μ) be a space equipped with a σ-algebra and a group Γ acting on M and preserving the σ-algebra, and $\mathcal M$ the cone of finite inivariant measures on M. Consider a finite, Γ-invariant measure on M. Then the following are equivalent.

(a) $\mu\in\mathcal M$ lies in the extremal ray of $\mathcal M$

(b) μ is ergodic.

(a) implies (b): Let U be an Γ-invariant measurable subset. Then $\mu=\mu|_U+\mu\Big|_{M\backslash U}$, and one of these two measures must vanish, because μ is extremal.

(b) implies (a): Let $\mu=\mu_1+\mu_2$ be a decomposition of the measure μ onto a sum of two invariant measures. Then $\mu\gg\mu_1$ and $\mu\gg\mu_2$ (Lemma 2), hence μ is proportional to μ_1 and μ_2 (Lemma 1). ■

REMARK: A probability measure μ **lies on an extremal ray if and only if it is extreme as a point in the convex set of all probability measures (prove it).**

Existence of ergodic measures (reminder)

To prove existence of ergodic measures, we use the following strategy:

1. Define topology on the space $\mathcal{M}$ of finite measures (**"measure topology"** or **"weak-$*$ topology"**) such that the space of probability measures is compact.

2. Use Krein-Milman theorem.

THEOREM: (Krein-Milman) Let $K \subset V$ be a compact, convex subset in a locally convex topological vector space. **Then K is the closure of the convex hull of the set of its extreme points.**

This theorem implies that any Γ-invariant finite measure is a limit of finite sums of ergodic measures.

Faces of compact convex sets

DEFINITION: Face of a convex set $A \subset V$ is a convex subset $F \subset A$ such that for all $x, y \in A$ whenever $\alpha x + (1-\alpha)y \in F$, $0 < \alpha < 1$, we have $x, y \in F$.

EXAMPLE: Let $A \subset V$ be a convex set, and $\lambda : V \longrightarrow \mathbb{R}$ a linear map. Consider the set $F_\lambda := \{a \in A \mid \lambda(a) = \sup_{x \in A} \lambda(x)\}$. **Then F_λ is a face of A.**

REMARK: Let $x, y \in V$ be distinct points in a topological vector space. Hahn-Banach theorem implies that **there exists a continuous linear functional $\lambda : V \longrightarrow \mathbb{R}$ such that $\lambda(x) \neq \lambda(y)$.**

COROLLARY: The set of extreme points of a compact convex subset $A \subset V$ is non-empty.

Proof: Indeed, from the above argument it follows that A has a non-trivial face, which is also compact and convex. Intersection of a chain of faces $F_1 \supsetneq F_2 \supsetneq F_3 ...$ is also a face, which is non-empty because all F_i are compact. Now, Zorn lemma implies that the smallest face is a point. ■

Krein-Milman theorem

THEOREM: Let $A \subset V$ be a compact convex subset a topological vector space. **Then A is the closure of the convex hull of the set $E(A)$ of extreme points of A.**

Proof: Let A_1 be the closure of the convex hull of the set $E(A)$ of extreme points of A. Suppose that $A_1 \subsetneq A$. Using Hahn-Banach theorem, we can find a λ which vanishes on A_1 and satisfies $\lambda(z) > 0$ for some $z \in A$. Then the face $F_\lambda = \{a \in A \mid \lambda(a) = \sup_{x \in A} \lambda(x)\}$ does not intersect A_1 and contains an extreme point, as shown above. ■

Choquet theorem

THEOREM: (Choquet theorem) Let $K \subset V$ be a compact, convex subset in a locally convex topological vector space, R the closure of the set $E(K)$ of its extreme points, and P the space of all probabilistic Borel measures on R. Consider the map $\Phi: P \longrightarrow K$ putting μ to $\int_{x\in R} x\mu$. **Then Φ is surjective.**

Proof: By weak-$*$ compactness of the space of measures, P is compact. The image of Φ is convex and contains all points of R which correspond to atomic measures. On the other hand, an image of a compact set under a continuous map is compact, hence $\Phi(P)$ is compact and complete. Finally, K is a completion of a convex hull of R, hence $K = \Phi(P)$. ■

REMARK: The measure μ associated with a point $k \in K$ is not necessarily unique. If $\Phi: P \longrightarrow K$ is bijective, the set K is called **a simplex**.

Ergodic decomposition of a measure

THEOREM: Let Γ be a group (or a semigroup) acting on a topological space M and preserving the Borel σ-algebra, P the space of all Γ-invariant probabilistic measures on M, and R the space of ergodic probabilistic measures. Then, for each $\mu \in P$, **there exists a probability measure ρ_μ on R, such that $\mu = \int x \in Rx\rho_\mu$.** Moreover, if Γ is countable, **the measure ρ_μ is uniquely determined by μ.**

REMARK: Such a form ρ_μ is called **ergodic decomposition** of a form μ.

Existence of ergodic decomposition follows from Choquet theorem. We prove uniqueness of ergodic decomposition in the next lecture.