Teoria Ergódica Diferenciável

lecture 17: Weak mixing on torus

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Convergence in density (reminder)

DEFINITION: The **(asymptotic) density** of a subset $J \subset \mathbb{Z}^{\geqslant 1}$ is the limit $\lim_N \frac{|J \cap [1,N]|}{N}$. A subset $J \subset \mathbb{Z}^{\geqslant 1}$ has **density 1** if $\lim_N \frac{|J \cap [1,N]|}{N} = 1$.

DEFINITION: A sequence $\{a_i\}$ of real numbers **converges to a in density** if there exists a subset $J \subset \mathbb{Z}^{\geqslant 1}$ of density 1 such that $\lim_{i \in J} a_i = a$. The convergence in density is denoted by $\operatorname{Dlim}_i a_i = a$.

PROPOSITION: **(Koopman-von Neumann, 1932)** Let $\{a_i\}$ be a sequence of bounded non-negative numbers, $a_i \in [0, C]$. **Then convergence to 0 in density is equivalent to the convergence of Cesàro sums:**

$$\operatorname{Dlim}_i a_i = 0 \Leftrightarrow \lim_N \frac{1}{N} \sum_{i=1}^{N} a_i = 0$$

Mixing, weak mixing, ergodicity (reminder)

DEFINITION: Let (M, μ, T) be a dynamic system, with μ a probability measure. We say that

(i) T **is ergodic** if $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}\mu(T^i(A)\cap B) = \mu(A)\mu(B)$, for all measurable sets $A, B \subset M$.

(ii) T **is weak mixing** if $\underset{i\to\infty}{\text{Dlim}}\, \mu(T^i(A)\cap B) = \mu(A)\mu(B)$.

(iii) T **is mixing**, or **strongly mixing** if $\lim_{i\to\infty} \mu(T^i(A)\cap B) = \mu(A)\mu(B)$.

REMARK: **The first condition is equivalent to the usual definition of ergodicity by the previous remark.** Indeed, from (usual) ergodicity it follows that $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i(\chi_A) = \mu(A)$, which gives $\lim_n \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i(\chi_A)\chi_B = \mu(A)\chi(B)$ and the integral of this function is precisely $\mu(A)\mu(B)$. Conversely, if $\lim_n \int (T^*)^i(\chi_A)\chi_B$ depends only on the measure of B, the function $\lim_n \int (T^*)^i(\chi_A)$ is constant, hence T is ergodic in the usual sense.

REMARK: Clearly, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) (the last implication follows because the density convergence implies the Cesàro convergence). ■

Mixing and weak mixing on the product (reminder)

DEFINITION: Let (M,μ,T) be a dynamical system. Consider the dynamical system $(M,\mu,T)^2 := (M\times M, \mu\times\mu, T\times T)$, where $\mu\times\mu$ is the product measure on $M\times M$, and $T\times T(x,y) = (T(x), T(y))$.

THEOREM: Let (M,μ,T) be a dynamical system, and $(M,\mu,T)^2$ its product with itself. **Then $(M,\mu,T)^2$ is (weak) mixing if and only if (M,μ,T) is (weak) mixing.**

Proof. Step 1: To simplify the notation, assume $\mu(M)=1$. To see that (weak) mixing on $(M,\mu,T)^2$ implies the (weak) mixing on (M,μ,T), we take the sets $A_1 := A\times M$ and $B_1 := B\times M$. Then $\mu(T^i(A_1)\cap B_1) = \mu(T^i(A)\cap B)$ and $\mu(A_1)\mu(B_1) = \mu(A)\mu(B)$, hence

$$\lim_i \mu(T^i(A_1)\cap B_1) = \mu(A_1)\mu(B_1)$$

implies

$$\lim_i \mu(T^i(A)\cap B) = \mu(A)\mu(B).$$

Mixing and weak mixing on the product 2 (reminder)

THEOREM: Let (M, μ, T) be a dynamical system, and $(M, \mu, T)^2$ its product with itself. **Then $(M, \mu, T)^2$ is (weak) mixing if and only (M, μ, T) is (weak) mixing.**

Step 2: Conversely, assume that (M, μ, T) is mixing. Since the subalgebra generated by cylindrical sets is dense in the algebra of measurable sets, it would suffice to show that $\lim_i \mu(T^i(A_1) \cap B_1) = \mu(A_1)\mu(B_1)$ where $A_1, B_1 \subset M^2$ are cylindrical. Write $A_1 = A \times A'$, $B_1 = B \times B'$. Then $\mu(T^n A_1 \cap B_1) = \Big(\mu(T^n A \cap B)\Big)\Big(\mu(T^n A' \cap B')\Big)$. The first of the terms in brackets converges to $\mu(A)\mu(B)$, the second to $\mu(A')\mu(B')$, giving

$$\lim_i \mu(T^i(A_1) \cap B_1) = \mu(A)\mu(B)\mu(A')\mu(B') = \mu(A_1)\mu(B_1).$$

■

REMARK: The same argument also proves that **ergodicity of $(M, \mu, T)^2$ implies ergodicity of (M, μ, T).** The converse implication is invalid even for a circle.

Ergodic measures which are not mixing (reminder)

REMARK: Let $L_\alpha: S^1 \longrightarrow S^1$ be a rotation with irrational angle α. In angle coordinates on $S^1 \times S^1$, the rotation $L_\alpha \times L_\alpha$ acts as $L_\alpha \times L_\alpha(x,y) = (x+\alpha, y+\alpha)$. Therefore, the closure of the orbit of (x,y) is always contained in the closed set $\{(a,b) \in S^1 \times S^1 \ \mid \ a-b = x-y\}$, and $L_\alpha \times L_\alpha$ **has no dense orbits**.

This gives the claim.

CLAIM: Irrational rotation of a circle is ergodic, but not weakly mixing.

Proof: Otherwise, $L_\alpha \times L_\alpha$ would be weak mixing, and hence ergodic, on $S^1 \times S^1$. $\blacksquare$

Weak mixing and non-constant eigenfunctions (reminder)

I am going to prove the following theorem.

Theorem 1: Let (M, μ, T) be a dynamical system. **Then the following are equivalent.**

(i) (M, μ, T) **is weakly mixing.**

(ii) The Koopman operator $T: L^2(M, \mu) \longrightarrow L^2(M, \mu)$ **has no non-constant eigenvectors.**

(iii) $(M, \mu, T)^2$ **is ergodic.**

Tensor product

DEFINITION: Let V, V' be vector spaces over k, and W a vector space freely generated by symbols $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subspace generated by combinations $av \otimes v' - v \otimes av'$, $a(v \otimes v') - (av) \otimes v'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$, where $a \in k$. Define **the tensor product** $V \otimes_k V'$ as a quotient vector space W/W_1.

PROPOSITION: ("Universal property of the tensor product") **For any vector spaces V, V', R, there is a natural identification $\operatorname{Hom}(V \otimes_k V', R) = \operatorname{Bil}(V \times V', R)$.**

DEFINITION: A **basis** in a vector space V is a subset $\{v_\alpha\} \subset V$ which is linearly independent and generates V.

CLAIM: Suppose that V, W are vector spaces (without topology), and $\{v_\alpha\}$, $\{w_\beta\}$ the bases (in Cauchy sense) in these spaces. **Then $\{v_\alpha \otimes w_\beta\}$ is a basis in $V \otimes W$.**

Proof: The natural map $\langle v_\alpha \otimes w_\beta \rangle \longrightarrow V \otimes W$ is by construction surjective and invertible by the Universal Property of the tensor product. ■

Tensor product and functions on a product

THEOREM: Let $C(M)$ be the space of functions $f: M \longrightarrow \mathbb{R}$, and $C(N)$ the spave of functions $f: N \longrightarrow \mathbb{R}$. Consider the natural map $\Psi: C(M) \otimes C(N) \longrightarrow C(M \times N)$. **Then Ψ is injective.**

Proof. Step 1: For N, M finite Ψ is an isomorphism. Indeed, for any $m \in M$ and $n \in N$, the tensor product product $\chi_m \otimes \chi_n$ of atomic functions χ_m and χ_n is mapped to $\chi_{(m,n)}$, hence Ψ is surjective, and it is injective because $\dim C(M) \otimes C(N) = |M||N| = \dim C(M \times N)$.

Step 2: For any linearly independent set of k functions $f_1, ..., f_k \in C(M)$, consider restriction of $f_1, ..., f_k$ to a finite subset $M_0 \subset M$. If there is a linear relation $\sum_i \lambda_i f_i \Big|_{M_0}$ for each finite subset, this linear relation is true on M. Therefore, **linearly independent functions remain linearly independent if restricted on a sufficiently big finite subset.**

Step 3: Let $\{f_\alpha\}$ be a basis in $C(M)$, $\{g_\alpha\}$ a basis in $C(N)$. Then $\{f_\alpha \otimes g_\alpha\}$ is a basis in $C(M) \otimes C(N)$, indexed by $\alpha \in A$, $\beta \in B$. Any vector $x \in C(M) \otimes C(N)$ takes form $x = \sum_{i \in A_0, j \in B_0} x_{ij} f_i \otimes g_j$, where $A_0 \subset A$, $B_0 \subset B$ are finite subsets. Then $x\Big|_{M_0 \times N_0}$ is non-zero for some finite subsets $M_0 \subset M$, $N_0 \subset N$ (Step 2). This implies that $\Psi(x)\Big|_{M_0 \times N_0}$ is also non-zero (Step 1). ■

Tensor product of Hilbert spaces

DEFINITION: Let H, H' be two Hilbert spaces. The tensor product $H \otimes H'$ has a natural scalar product which is non-complete. Its completion $H \hat{\otimes} H'$ is called **completed tensor product** of H and H'.

REMARK: Let $\{e_i\}, \{e'_i\}$ be orthonormal bases in H, H'. **Then $H \hat{\otimes} H'$ is all series $\sum_i \alpha_{ij} e_i \otimes e'_j$ with $\sum_{i,j} |\alpha_{ij}|^2 < \infty$.**

CLAIM: Let (M, μ) and (M', μ') be metrizable spaces with Borel measure. **Then $L^2(M \times M', \mu \times \mu') = L^2(M, \mu) \hat{\otimes} L^2(M', \mu')$.**

Proof: The product map $L^2(M, \mu) \otimes L^2(M', \mu') \longrightarrow L^2(M \times M', \mu \times \mu')$ is injective because it it is injective on all functions, as shown above.

The tensor product $C^0(M) \otimes C^0(M')$ is a dense (by Stone-Weierstrass) subring in $C^0(M \times M)$, the space $L^2(M, \mu) \otimes L^2(M', \mu')$ is its partial completion, and $L^2(M, \mu) \hat{\otimes} L^2(M', \mu')$ is its completion. Therefore, $L^2(M, \mu) \otimes L^2(M', \mu') \subset L^2(M \times M', \mu \times \mu')$ is a dense subset. Therefore, both spaces $L^2(M, \mu) \hat{\otimes} L^2(M', \mu')$ and $L^2(M \times M', \mu \times \mu')$ are obtained as completions of $L^2(M, \mu) \otimes L^2(M', \mu')$. They are isomorphic because completion is unique. ■

Orthogonal operators on tensor square (reminder)

Last lecture we proved the following theorem.

THEOREM: Let U be an orthogonal operator on a Hilbert space H. **Then the following are equivalent:**

(i) U has no eigenvectors in H.

(ii) $U \times U$ has no eigenvectors in $H \hat{\otimes} H$ with eigenvalue 1.

This immediately implies equivalence of (ii) and (iii) in Theorem 1:

PROPOSITION: Let (M, μ, T) be a dynamical system. Then **$T \times T$ is ergodic on M^2 if and only if T has no non-constant eigenfunctions on $L^2(M, \mu)$.**

Proof: Let $H \subset L^2(M, \mu)$ be the space of all functions f with $\int_M f\mu = 0$. Then $L^2(M, \mu) = H \oplus \mathbb{R}$ and

$$L^2(M^2, \mu^2) = (H \oplus \mathbb{R}) \hat{\otimes} (H \oplus \mathbb{R}) = H \hat{\otimes} H \oplus H \oplus H \oplus \mathbb{R}.$$

Ergodicity of $T \times T$ on M^2 (and, hence, M) means that $T \times T$ has no invariant vectors in H and $H \otimes H$. **By the previous theorem, this is equivalent to T having no eigenvectors in H.** ■

Weak mixing and action on the square

Theorem 1: Let (M, μ, T) be a dynamical system. **Then the following are equivalent.**

(i) (M, μ, T) **is weakly mixing.**

(ii) The Koopman operator $T: \ L^2(M, \mu) \longrightarrow L^2(M, \mu)$ **has no non-constant eigenvectors.**

(iii) $(M, \mu, T)^2$ **is ergodic.**

Proof. Step 1: Equivalence of (iii) and (ii) is already proven. Implication (i) $\Rightarrow$ (iii) is elementary: indeed, $(M, \mu, T)^2$ is weakly mixing, hence ergodic. It remains only to prove that (iii) implies (i).

Weak mixing and action on the square (2)

Ergodicity of $(M,\mu,T)^2$ imlplies that (M,μ,T) is weak mixing:

Step 2: Let $A,B\subset M$ be measurable subsets. To simplify notation, we assume that $\mu(M)=1$. Consider the sequence $\frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^iA\cap B)\mu(M)-\mu(A)\mu(B))^2$. The terms are non-negative, and by Koopman-von Neumann **convergence of this sequence implies density convergence of $\mu(T^iA\cap B)-\mu(A)\mu(B)$, which is the same as weak mixing.**

Step 3:

$$\frac{1}{n}\sum_{i=0}^{n-1}(\mu(T^iA\cap B)-\mu(A)\mu(B))^2=\left[\frac{1}{n}\sum_{i=0}^{n-1}\mu(T^iA\cap B)^2-\mu(A)^2\mu(B)^2\right]$$
$$+\left[\frac{2}{n}\sum_{i=0}^{n-1}\mu(A)^2\mu(B)^2-\mu(T^iA\cap B)\mu(A)\mu(B).\right]$$

The first term on RHS is $\frac{1}{n}\sum_{i=0}^{n-1}\mu((T\times T)^iA^2\cap B^2)-\mu(A^2)\mu(B^2)$, and it converges because $T\times T$ is ergodic. The second term is

$$-\mu(A)\mu(B)\frac{2}{n}\sum_{i=0}^{n-1}\mu(T^iA\cap B)-\mu(A)\mu(B),$$

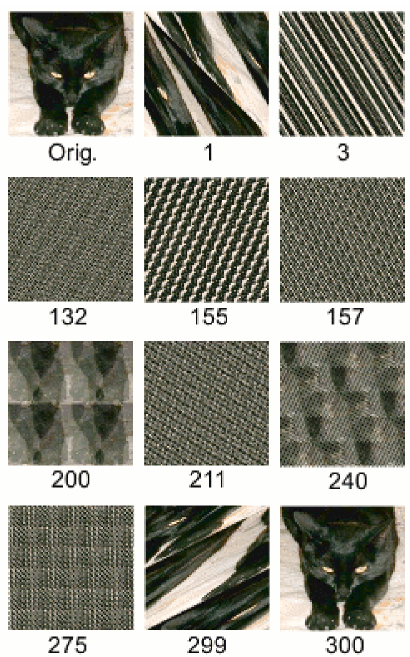
and it converges because M is ergodic. ∎

Arnold's cat map (reminder)

DEFINITION: The Arnold's cat map is $A: \ T^2 \longrightarrow T^2$ defined by $A \in SL(2,\mathbb{Z})$,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues of A are roots of $\det(t\,\mathrm{Id} - A) = (t-2)(t-1) - 1 = t^2 - 3t - 1$. This is a quadratic equation with roots $\alpha_\pm = \frac{3 \pm \sqrt{5}}{2}$. On the vectors tangent to the eigenspace of α_-, the map A^n acts as $(\alpha_-)^n$, hence the stable foliation is tangent to these vectors. Similarly, unstable foliation is tangent to the eigenspace of α_+. **This map is ergodic by E. Hopf theorem.**



Weak mixing for a torus

THEOREM: Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be a torus, and $A \in SL(n,\mathbb{Z})$ a matrix which has no vectors $v \neq 0$ with $\{A^i(v), i = 0,1,2,...,\}$ finite. **Then the action of A on T^n is weak mixing.**

Proof. Step 1: Let $t_1,...,t_n$ be coordinates on $\mathbb{R}^n$. We can think of t_i as of angle coordinates on T^n. Consider the Fourier monomials $F_{l_1,...,l_n} := e^{2\pi\sqrt{-1}\sum l_i t_i}$, where $l_1,...,l_n$ are integers. As shown above,

$$L^2(T^n) \cong \underbrace{L^2(S^1)\hat{\otimes}L^2(S^1)\otimes ...L^2(S^1)}_{n \text{ times}}.$$

This implies that the **Fourier monomials form a Hilbert basis in $L^2(T^n)$.**

Step 2: Let $f \in L^2(T^n)$ be an eigenvector of A, and $f = \sum \alpha_{l_1,...,l_n} F_{l_1,...,l_n}$ its Fourier decomposition. Consider the set S_ε of all n-tuples $l_1,...,l_n \in \mathbb{Z}^n$ such that $|\alpha_{l_1,...,l_n}| > \varepsilon$. Since $\sum |\alpha_{l_1,...,l_n}|^2 < \infty$, the set S_ε is finite for all $\varepsilon > 0$. Since f is an eigenfunction of A, and A is unitary on $L^2(T^n)$, one has $A(f) = uf$, with $|u| = 1$, and **S_ε is A-invariant.** This is impossible, unless $S_\varepsilon = \{(0,0,...,0)\}$, because A acts on all non-zero vectors with infinite orbits.

Step 3: We proved that A **has no non-constant eigenfunctions on $L^2(T^n)$,** hence it is weak mixing. ■