Teoria Ergódica Diferenciável

lecture 13: Hopf theorem on geodesic flows

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Natural parametrization (reminder)

DEFINITION: Let $\gamma : [a,b] \longrightarrow M$ be a path, and $\psi : [a,b] \longrightarrow [c,d]$ **Parametrization** of the path γ is the map $\psi \circ \gamma : [c,d] \longrightarrow M$, the same path parametrized differently. **Natural parametrization** of a minimizing geodesic γ, $L(\gamma) = a$ is parametrization $\gamma : [0,a] \longrightarrow M$ such that the length of $\gamma\Big|_{[0,t]}$ is equal t. Clearly, $\gamma\Big|_{[0,t]} = t$ defines the parametrization of γ uniquely.

REMARK: Let $\gamma : [0,a] \longrightarrow M$ be a minimizing geodesic with natural parametrization. **Then γ is an isometric embedding.**

DEFINITION: A geodesic $\gamma : [a,b] \longrightarrow M$ **has natural parametrization** if γ is locally an isometry.

THEOREM: Let M be a Riemannian manifold, $x \in M$ and $v \in T_xM$ be a tangent vector. **Then there exists a unique geodesic $\gamma : [0,a] \longrightarrow M$ with natural parametrization such that $\gamma(0) = x$ and $\gamma'(0) = v$.** Moreover, the map γ smoothly depends on x and v.

Proof: We proved this theorem for the hyperbolic space; for Euclidean metric it is well known. **The proof for a more general Riemannian manifold is left as an exercise.** ∎

The exponential map (reminder)

DEFINITION: Let M be a Riemannian manifold. For any $v \in T_xM$ with $|v| = 1$, denote the corresponding naturally parametrized geodesic by $t \longrightarrow \exp(tv)$. The map $T_xM \longrightarrow M$ mapping $v \in T_xM$ to $\exp\left(|v|\frac{v}{|v|}\right)$ is called **the exponential map**.

THEOREM: **Exponential map is a diffeomorphism for $|v|$ sufficiently small.**

Proof: Again, for Euclidean and hyperbolic space this theorem is proven, and for an arbitrary Riemannian manifold it is left as an exercise. ■

Geodesic flow (reminder)

DEFINITION: Let M be a manifold. **Spherical tangent bundle** $SM \subset TM$ is the space of all tangent vectors of length 1.

DEFINITION: Consider the map

$$\Psi_t(v, x) = (\exp(tv), d\exp(tv)(v))$$

mapping $v \in T_xM, t \in \mathbb{R}$ to $d\exp(tv)(v)) \in T_{\exp(tv)}M$; here

$$d\exp(tv): \ T_xM \longrightarrow T_{\exp(tv)}M$$

is the differential of the exponent map $\exp: \ T_xM \longrightarrow M$. This defines an action of $\mathbb{R}$ on SM, $t \longrightarrow \Psi_t \in \operatorname{Diff}(SM)$. This action is called **the geodesic flow**.

REMARK: Geodesic flow **takes a unit tangent vector, takes a naturally parametrized geodesic tangent to this vector, and moves this vector along this geodesic.**

Volume forms

DEFINITION: Grassmann algebra is an algebra $\Lambda^*(V^*)$ of a vector space V is an algebra of antisymmetric k-forms on V (similar to polynomial, but antisymmetric instead of symmetric).

THEOREM: Let $x_1, ..., x_n$ be a basis in V^*. Then **the space $\Lambda^k V^*$ is generated by antisymmetric forms $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_k}$, $i_1 < i_2 < ... < i_k$, which are all linearly independent.**

COROLLARY: The space $\Lambda^k(V^*)$ of k-linear antisymmetric forms is $\binom{n}{k}$-dimensional, where $n = \dim V$. In particular, **$\Lambda^n(V^*)$ is 1-dimensional.**

DEFINITION: The space of volume forms on an n-dimensional vector space V is $\Lambda^n(V^*)$.

DEFINITION: Orientation on V is a choice of positive direction on $\Lambda^n(V^*)$. A **positive volume form** on an oriented vector space V is a volume form which is positive in the sense of orientation.

REMARK: There is a bijection between translation invariant Lebesgue measures on $\mathbb{R}^n$ and positive volume forms on $\mathbb{R}^n$.

Differential forms

DEFINITION: Let M be a manifold. A **differential form**, or **k-form**, on M is a choice of a vector $\lambda_x \in \Lambda^k(T_x^*M)$ smoothly depending on x. The space of all differential forms on M is denoted $\Lambda^k M$.

DEFINITION: A positive volume form on an oriented n-manifold is a differential form $\nu \in \Lambda^n M$ such that at each $x \in M$, ν defines a positive volume form on $\Lambda^n(T_x^*M)$.

REMARK: Let $f: M \longrightarrow N$ be a smooth map of manifolds. Then f defines a **pullback map** $f^*: \Lambda^k N \longrightarrow \Lambda^k M$ which takes a form $\eta \in \Lambda^k N$ and puts it to

$$f^*(\eta)(v_1, ..., v_k) = \eta(D_f(v_1), D_f(v_2), ..., D_f(v_k)).$$

Integral and measure associated with a differential form

DEFINITION: Let M be an oriented n-dimensional manifold, and $\Lambda^n_c M$ the space of volume forms with compact support. **Integral** is a linear map $\int_M : \Lambda^n_c M \longrightarrow \mathbb{R}$ which satisfies the following conditions.

(invariance) For any diffeomorphism $f: M \longrightarrow M$, and any $\nu \in \Lambda^n_c M$, one has $\int_M \nu = \int_M f^*\nu$.

(positivity) For any non-negative volume form ν, one has $\int_M \nu \geqslant 0$.

THEOREM: The functional $\int_M : \Lambda^n M \longrightarrow \mathbb{R}$ satisfying these properties **exists and is unique.**

This theorem is left as an exercise.

Riesz representation theorem: Let M be a metrizable, locally compact topological space, and $C^0_c(M)^*$ the space of functionals continuous in uniform topology. **Then Radon measures can be characterized as continuous functionals $\mu \in C^0_c(M)^*$ which are non-negative on all non-negative functions.**

Proof: Lecture 6.

DEFINITION: For any volume form ν on M define a Radon measure ν associated to the functional $w \longrightarrow \int_M w\nu$. This measure is called **Lebesgue measure associated with a differential form**.

Smooth measures

DEFINITION: Let μ be a signed measure on a smooth manifold M. We say that μ is of class C^1 if for any vector field $X \in TM$ there exists a signed measure $\operatorname{Lie}_X \mu$ such that $\int_M D_X(f)\mu = \int_M f \operatorname{Lie}_x \mu$. We say that **$\mu$ is of class C^i** if it is of class C^1 and $\operatorname{Lie}_X \mu$ is of class C^{i-1} for any vector field X, and **smooth** (or **of class C^∞**) if it is of class i for all $i > 0$.

THEOREM: A signed measure μ on an n-manifold M is of class C^i, $i > 0$, **if and only if it is associated with a differential form $\nu \in \Lambda^n M$ of class C^i.**

This theorem is left as an exercise.

Spherical bundle for a space form

REMARK: Let $M = G/H$ be a homogeneous space. Then a G-invariant volume form on M is unique up to a constant. Indeed, we can take the volume form in a given tangent space and extend it to a G-invariant volume by G-action; thus, **a volume form on T_xM determines the measure on M.**

CLAIM: Let $M = G/H$ be a space form. **Then the natural action of G on SM is transitive.**

Proof: G acts transitively on M, and $H = SO(n)$ acts transitively on the sphere $\{v \in T_xM \ \mid \ |v| = 1\}$. ∎

REMARK: This also implies that $SM = G/SO(n-1)$. Indeed, $\operatorname{St}_{SO(n)}(v) = SO(n-1)$.

Riemannian volume and geodesic flow

THEOREM: Let $M = G/H$ be a space form, SM its spherical bundle, and Vol a G-invariant volume form. **Then the geodesic flow preserves Vol.**

Proof. Step 1: Since the geodesic flow Ψ_t is G-equivariant, the map $t \longrightarrow (\Psi_t)_* \mathrm{Vol} = \lambda_t \mathrm{Vol}$ defines an action of $\mathbb{R}$ on the 1-dimensional space of G-invariant volume forms, that is, a homomorhism $\mathbb{R} \longrightarrow \mathbb{R}^*$. This gives $\lambda_{-t} = \lambda_t^{-1}$.

Step 2: Let $\tau_x : M \longrightarrow M$ be the central symmetry with center in $x \in M$. Then $\Psi_t \circ \tau\Big|_{T_x SM} = \tau \circ \Psi_{-t}\Big|_{T_x SM}$ because the central symmetry reverses the orientation on geodesics. Then $\lambda_t = \lambda_{-t}$. ■

Hopf argument

DEFINITION: Let M be a metric space with a Borel measure and $F_t : M \times \mathbb{R} \longrightarrow M$ a continuous flow preserving measure. The **"stable foliation"** is an equivalence relation on M, with $x \sim y$ when $\lim_{t\mapsto\infty} d(F_t(x), F_t(y)) = 0$. The **"leaves"** of the stable foliation are the equivalence classes. **Unstable foliation** is the stable foliation for F_{-t}.

THEOREM: (Hopf Argument) Any measurable, F_t-invariant function **is constant on the leaves of the stable foliation** outside of a measure 0 set.

Proof: Lecture 7. ■

DEFINITION: We say that M is **a Riemannian manifold of constant negative curvature** if it is locally isometric to a hyperbolic space.

THEOREM: (E. Hopf) Let M be a complete Riemannian manifold of finite volume and constant negative curvature. **Then the geodesic flow is ergodic.**

Proof (for dimension 2) is later in this lecture; it remains as an exercise to extend this proof to any dimension.

Absolute

Let $V=\mathbb{R}^3$ be a vector space with bilinear form of signature (1,2). Denote by V^+ the positive cone of V, that is, one of two connected components of $\{v\in V \ \mid \ (v,v)>0\}$. Consider the hyperbolic space $\mathbb{H}=SO^+(1,2)/SO(2)$ as projectivization of V^+, $\mathbb{H}=\mathbb{P}V^+=V^+/\mathbb{R}^{>0}$. Let $\overline{\mathbb{H}}$ be the closure of $\mathbb{P}V^+\subset\mathbb{P}V=\mathbb{R}P^2$.

DEFINITION: The infinite circle $\partial\Delta$ considered as a boundary of the disk $\mathbb{P}V^+=\mathbb{H}$ is called **the absolute** of the projective plane.

REMARK: Any isometry of the disk is naturally extended to the absolute. Indeed, $SO^+(1,2)$ acts on the real projective space $\mathbb{R}P^2$, and absolute is the boundary of $\mathbb{P}V^+$ in $\mathbb{R}P^2$.

Convergence of geodesics

REMARK: From now on, **all geodesics are considered with their natural parametrization.**

REMARK: From the description of geodesics in Poincare disc, it is clear that for any geodesic $\gamma :]\infty, \infty[\longrightarrow \Delta$ the limit points $\gamma_+ := \lim_{t\mapsto\infty} \gamma(t)$ and $\gamma_- := \lim_{t\mapsto-\infty} \gamma(t)$ are well defined in the absolute $\partial\Delta$, and, moreover, the points $\gamma_+, \gamma_- \in \partial\Delta$ determine the geodesic uniquely.

REMARK: The Poincaré metric on $\mathbb{H}$ is $d_P = \frac{dx^2+dy^2}{y^2}$. Therefore,

$$\lim_{u\longrightarrow\infty} d_P((t_1, u_1+u), (t_2, u_2+u)) = 0.$$

This gives the following

COROLLARY: Let γ, δ be geodesics such that their $+\infty$-limits $\gamma_+, \delta_+ \in \partial\Delta$ are equal, and $t_1 \in \mathbb{R}$ any number. **Then there exists $t_2 \in \mathbb{R}$ such that the tangent vectors $\dot\gamma(t_1), \dot\delta(t_2) \in S\Delta$ belong to the same leaf of the stable foliation.** ■

Hopf theorem for manifolds of constant negative curvature

THEOREM: (E. Hopf) Let M be a complete 2-dimensional Riemannian manifold of finite volume and constant negative curvature. **Then the geodesic flow Ψ_t is ergodic.**

Proof. Step 1: Any such M **is obtained as a quotient** of the hyperbolic plane Δ/Γ, where Γ is a discrete group acting on Δ by isometries.

Step 2: To prove Hopf Theorem it would suffice to show that a function which is (*) constant on orbits of the geodesic flow and on almost all leaves of stable foliation on $S\Delta$ and (**) on orbits of the geodesic flow and on almost all leaves of unstable foliation is necessarily constant. This follows from the Hopf argument (Lecture 7).

Step 3: For (*) f should be constant on all S_α, where S_α is all $v \in T_x\Delta$ such that the geodesic tangent to v end up in a point $\alpha \in \partial\Delta$. For (**), f should be constant on all U_β, where U_β is all vectors $v \in T_x\Delta$ such that the geodesic tangent to v begins in $\beta \in \partial\Delta$. **The sets S_α, U_β intersect in a geodesic connecting α to β** which exists whenever $\alpha \neq \beta$. ■