Teoria Ergódica Diferenciável

lecture 12: Geodesic flow

Instituto Nacional de Matemática Pura e Aplicada

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Upper half-plane (reminder)

REMARK: The map $z \longrightarrow -\sqrt{-1}\,(z-1)^{-1}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}$.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ acts on the upper half-plane $\mathbb{H}$ as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

Proof: The group $PSL(2,\mathbb{R})$ preserves the line $\operatorname{im} z = 0$, hence acts on $\mathbb{H}$ by conformal automorphisms. The stabilizer of a point is S^1 **(prove it).** Now, Lemma 2 implies that $PSL(2,\mathbb{R}) = PU(1,1)$. ■

REMARK: We have shown that $\mathbb{H} = SO(1,2)/S^1$, hence **$\mathbb{H}$ is conformally equivalent to the hyperbolic space.**

Upper half-plane as a Riemannian manifold (reminder)

DEFINITION: Poincaré half-plane is the upper half-plane equipped with an $PSL(2,\mathbb{R})$-invariant metric. By constructtion, **t is isometric to the Poincare disk and to the hyperbolic space form.**

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}$. **Then the Riemannian structure s on $\mathbb{H}$ is written as $s = \operatorname{const}\frac{dx^2+dy^2}{y^2}$.**

Proof: Since the complex structure on $\mathbb{H}$ is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2+dy^2)$, where $\mu \in C^\infty(\mathbb{H})$. **It remains to find μ, using the fact that s is $PSL(2,\mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $x \longrightarrow x + a$ fixes s, hence μ is a function of y. For any $\lambda \in \mathbb{R}^{>0}$, the map $H_\lambda(x) = \lambda x$, being holomorphic, also fixes s; since $\mathbb{H}_\lambda(dx^2 + dy^2) = \lambda^2 dx^2 + dy^2$, we have $\mu(\lambda x) = \lambda^{-2}\mu(x)$. ■

Geodesics on Riemannian manifold (reminder)

DEFINITION: Minimising geodesic in a Riemannian manifold is a piecewise smooth path connecting x to y such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path γ such that for any $x \in \gamma$ there exists a neighbourhood of x in γ which is a minimising geodesic.

EXERCISE: Prove that a big circle in a sphere is a geodesic. Prove that an interval of a big circle of length $\leqslant \pi$ is a minimising geodesic.

REMARK: Further on, **all Riemannian manifold are tacitly assumed to be complete** with respect to the geodesic distance.

Geodesics in Poincaré half-plane (reminder)

THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2,\mathbb{R})$.

Proof. Step 1: Let $a, b \in \mathbb{H}$ be two points satisfying $\operatorname{Re} a = \operatorname{Re} b$, and l the line connecting these two points. Denote by Π the orthogonal projection from $\mathbb{H}$ to the vertical line connecting a to b. For any tangent vector $v \in T_z\mathbb{H}$, one has $|D\pi(v)| \leqslant |v|$, and the equality means that v is vertical (prove it). Therefore, **a projection of a path γ connecting a to b to l has length $\leqslant L(\gamma)$, and the equality is realized only if γ is a straight vertical interval.**

Step 2: For any points a, b in the Poincaré half-plane, **there exists an isometry mapping (a, b) to a pair of points (a_1, b_1) such that $\operatorname{Re}(a_1) = \operatorname{Re}(b_1)$. (Prove it!)**

Step 3: Using Step 2, we prove that **any geodesic γ on a Poincaré half-plane is obtained as an isometric image of a straight vertical line:** $\gamma = v(\gamma_0)$, $v \in \operatorname{Iso}(\mathbb{H}) = PSL(2,\mathbb{R})$ ∎

Geodesics in Poincaré half-plane (reminder)

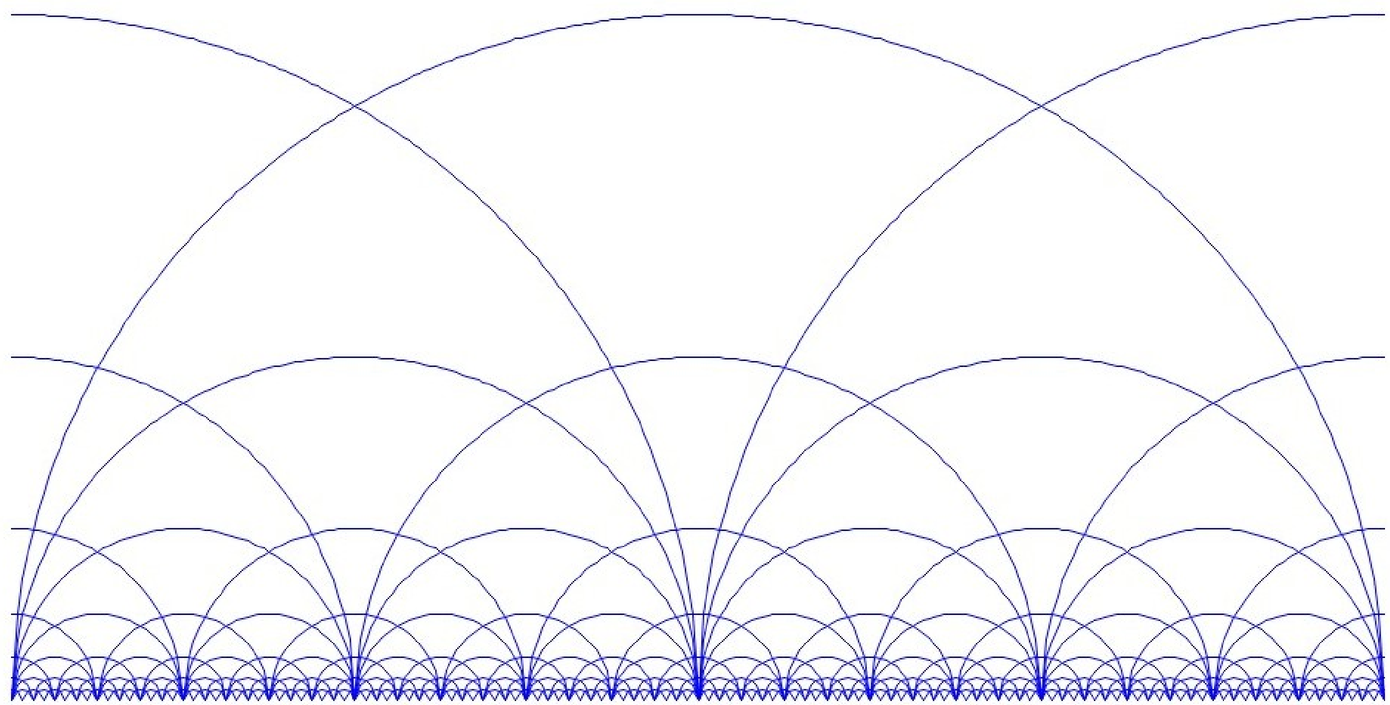
CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and $S_1$the closure of its image in $\mathbb{C}P^1 \subset \mathbb{C}$. Here $\mathbb{C}$ is embedded to $\mathbb{C}P^1$ by the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ■

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

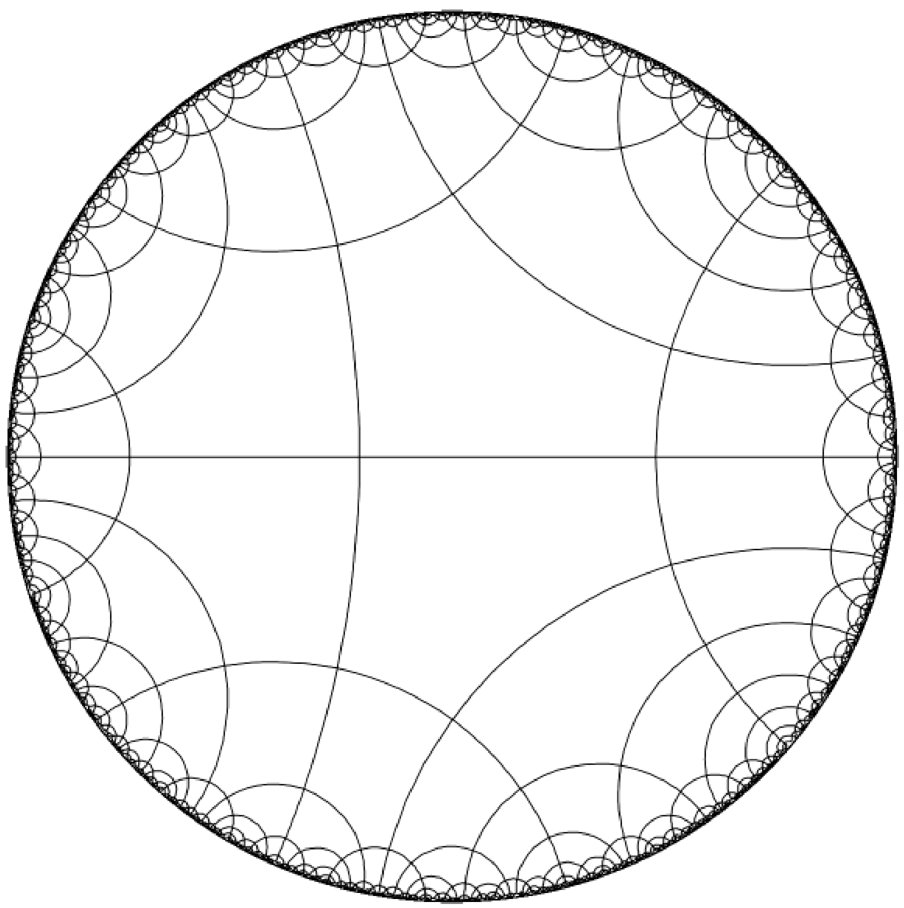
Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■

Geodesics on Poincare half-plane

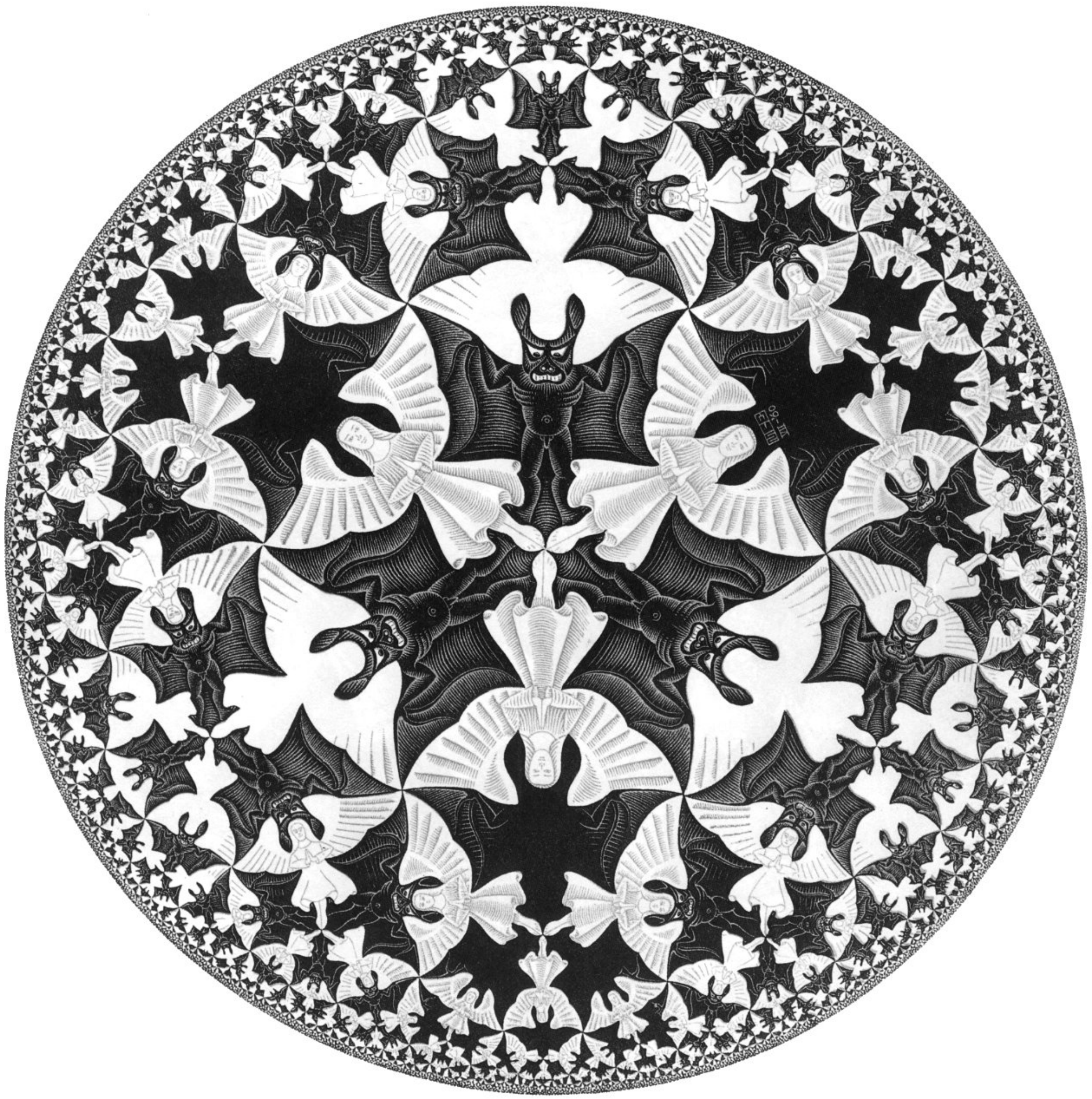


Geodesics on Poincare disc

REMARK: Geodesics on Poincare disc are half-circles orthogonal to its boundary. Indeed, Poincare disc is obtained from Poincare plane by a Möbius transform, and Möbius transforms preserve map circles and lines to circles and lines.



Maurits Cornelis Escher, Circle Limit IV (1960)



Maurits Cornelis Escher, Circle Limit V (1960)

Natural parametrization

DEFINITION: Let $\gamma: [a,b] \longrightarrow M$ be a path, and $\psi: [a,b] \longrightarrow [c,d]$ **Parametrization** of the path γ is the map $\psi \circ \gamma: [c,d] \longrightarrow M$, the same path parametrized differently. **Natural parametrization** of a minimizing geodesic γ, $L(\gamma) = a$ is parametrization $\gamma: [0,a] \longrightarrow M$ such that the length of $\gamma\Big|_{[0,t]}$ is equal t. Clearly, $\gamma\Big|_{[0,t]} = t$ defines the parametrization of γ uniquely.

REMARK: Let $\gamma: [0,a] \longrightarrow M$ be a minimizing geodesic with natural parametrization. **Then γ is an isometric embedding.**

DEFINITION: A geodesic $\gamma: [a,b] \longrightarrow M$ **has natural parametrization** if γ is locally an isometry.

THEOREM: Let M be a Riemannian manifold, $x \in M$ and $v \in T_xM$ be a tangent vector. **Then there exists a unique geodesic $\gamma: [0,a] \longrightarrow M$ with natural parametrization such that $\gamma(0) = x$ and $\gamma'(0) = v$.** Moreover, the map γ smoothly depends on x and v.

Proof: We proved this theorem for the hyperbolic space; for Euclidean metric it is well known. **The proof for a more general Riemannian manifold is left as an exercise.** ■

The exponential map

DEFINITION: Let M be a Riemannian manifold. For any $v \in T_xM$ with $|v| = 1$, denote the corresponding naturally parametrized geodesic by $t \longrightarrow \exp(tv)$. The map $T_xM \longrightarrow M$ mapping $v \in T_xM$ to $\exp\left(|v|\frac{v}{|v|}\right)$ is called **the exponential map**.

THEOREM: Exponential map is a diffeomorphism for $|v|$ sufficiently small.

Proof: Again, for Euclidean and hyperbolic space this theorem is proven, and for an arbitrary Riemannian manifold it is left as an exercise. ■

Geodesic flow

DEFINITION: Let M be a manifold. **Spherical tangent bundle** $SM \subset TM$ is the space of all tangent vectors of length 1.

DEFINITION: Consider the map

$$\Psi_t(v,x) = (\exp(tv), d\exp(tv)(v))$$

mapping $v \in T_xM, t \in \mathbb{R}$ to $d\exp(tv)(v)) \in T_{\exp(tv)}M$; here

$$d\exp(tv): \ T_xM \longrightarrow T_{\exp(tv)}M$$

is the differential of the exponent map $\exp: \ T_xM \longrightarrow M$. This defines an action of $\mathbb{R}$ on SM, $t \longrightarrow \Psi_t \in \operatorname{Diff}(SM)$. This action is called **the geodesic flow**.

REMARK: Geodesic flow **takes a unit tangent vector, takes a naturally parametrized geodesic tangent to this vector, and moves this vector along this geodesic.**

Riemannian volume

DEFINITION: Let M be an n-dimensional Riemannian manifold. Define **the Riemannian volume** as a measure which sets the volume of a very small n-cube with sides $\varepsilon + o(\varepsilon)$ to $\varepsilon^n + o(\varepsilon)$.

DEFINITION: Let M be a manifold. It takes some work to define the Riemannian structure on SM. However, for M Euclidean or hyperbolic, SM is homogeneous, and we can take any metric at a point, average it with respect to the isotropy group (which is compact, because it is contained in $SO(n-1)$, which is the stabilizer of a point of M), and extend the averaged metric to SM by homogeneity. **This defines a G-invariant Lebesgue measure on SM,** where $M = G/H$ is a space form. This measure is called **the Liouville measure**.

THEOREM: Geodesic flow preserves the Liouville measure on SM.

For M arbitrary this theorem takes lots of work, for M a space form we prove it in the next slide.

Riemannian volume and geodesic flow

REMARK: Let $M = G/H$ be a homogeneous space. Then a G-invariant volume form on M is unique up to a constant. Indeed, we can take the volume form in a given tangent space and extend it to a G-invariant volume by G-action; thus, **a volume form on T_xM determines the measure on M.**

THEOREM: Let $M = G/H$ be a space form, SM its spherical bundle and Vol a G-invariant volume form. **Then the geodesic flow preserves Vol.**

Proof: Since the geodesic flow Ψ_t is G-equivariant, the map $t \longrightarrow (\Psi_t)_* \operatorname{Vol} = \lambda_t \operatorname{Vol}$ defines an action of $\mathbb{R}$ on the 1-dimensional space of G-invariant volume forms, that is, a homomorhism $\mathbb{R} \longrightarrow \mathbb{R}^*$. This gives $\lambda_t\lambda_{-t} = 1$. However, Ψ_t is conjugate to Ψ_{-t} via central symmetry. Therefore, $\lambda_t = \lambda_{-t} = 1$. ■