Teoria Ergódica Diferenciável

lecture 10: Group of conformal automorphisms

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Riemannian manifolds (reminder)

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x, y \in M$, and any piecewise smooth path $\gamma : [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines a metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Conformal structures (reminder)

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

DEFINITION: **A Riemann surface** is a 2-dimensional oriented manifold equipped with a conformal structure.

DEFINITION: Let $I: TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

CLAIM: Let M be a 2-dimensional oriented conformal manifold. **Then M admits a unique orthogonal almost complex structure** in such a way that the pair $x, I(x)$ is positively oriented. Conversely, **an almost complex structure uniquely determines the conformal structure nd orientation.**

Homogeneous spaces (reminder)

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation (reminder)

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A Riemannian form Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant scalar product on T_xM. For any $y \in M$ obtained as $y = g(x)$, consider the form Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ_x is h-invariant, **the metric Φ_y is independent from the choice of g.**

We proved

THEOREM: **Homogeneous Riemannian forms on $M = G/H$ are in bijective correspondence with isotropy invariant spalar products on T_xM,** for any $x \in M$. ■

Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/O(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms (reminder)

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: From now on, all space forms are assumed to be homogeneous Riemannian manifolds.

Schwartz lemma (reminder)

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

EXERCISE: Prove the maximum principle.

LEMMA: (Schwartz lemma) Let $f: \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = \text{const}$. ■

Conformal automorphisms of the disk act transitively

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group** $\operatorname{Aut}(\Delta)$ **of its holomorphic automorphisms acts on** Δ **transitively.**

Proof. Step 1: Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$.

Step 2: For $|z| = 1$, we have

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1.$$

Therefore, V_a preserves the circle. Maximum principle implies that V_a maps its interior to its interior.

Step 3: To prove invertibility, we interpret V_a as an element of $PGL(2, \mathbb{C})$. ■

Transitive action is determined by a stabilizer of a point

Lemma 1: Let $M = G/H$ be a homogeneous space, and $\Psi: G_1 \longrightarrow G$ a homomorphism such that G_1 acts on M transitively and $\mathrm{St}_x(G_1) = \mathrm{St}_x(G)$. **Then $G_1 = G$.**

Proof: Since any element in $\ker\Psi$ belongs to $\mathrm{St}_x(G_1) = \mathrm{St}_x(G) \subset G$, the homomorphism Ψ is injective. It remais only to show that Ψ is surjective.

Let $g \in G$. Since G_1 acts on M transitively, $gg_1(x) = x$ for some $g_1 \in G_1$. Then $gg_1 \in \mathrm{St}_x(G_1) = \mathrm{St}_x(G) \subset \operatorname{im} G_1$. This gives $g \in G_1$. ■

Group of conformal automorphisms of the disk is $PU(1,1)$

REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \ | \ h(l,l) > 0\}$ by holomorphic automorphisms. Indeed, $PGL(2,\mathbb{C})$ acts conformally on $\mathbb{C}P^1$.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\operatorname{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi : \ PU(1,1) \longrightarrow \operatorname{Aut}(\Delta)$ the map constructed above. **Then Ψ is a group isomorphism.**

Proof: We use Lemma 1. Both groups act on Δ transitively, hence **it suffices only to check that $\operatorname{St}_x(PU(1,1)) = S^1$ and $\operatorname{St}_x(\operatorname{Aut}(\Delta)) = S^1$.** The first isomorphism is clear, because the space of unitary automorphisms fixing a vector v is $U(v^\perp)$. The second isomorphism follows from Schwartz lemma **(prove it!)**. ■

Conformal automorphism and the Poincare metric on the disc

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then (Δ, h) is conformally equivalent to $(\Delta, \text{flat metric})$.**

Proof: The group $\operatorname{Aut}(\Delta) = PU(1,1)$ acts on Δ holomorphically, that is, preserving the conformal structure of the flat metric. However, the homogeneous conformal structure on $PU(1,1)/S^1$ is unique for the same reason the homogeneous metric is unique up to a contant multiplier **(prove it)**. ■

COROLLARY: All **conformal automorphisms of a disk are isometries.**

Proof: The group $\operatorname{Aut}(\Delta)$ acts by homotheties, because an $\operatorname{Aut}(\Delta)$-invariant metric on a space G/S^1 is unique up to homothety. However, a homothety of a disk is an isometry by Schwartz lemma. ■

DEFINITION: **Poincare metric** on a disc $\Delta \subset \mathbb{C}$ is any $\operatorname{Aut}(\Delta)$-invariant metric, where $\operatorname{Aut}(\Delta)$ is the group of conformal isometries.

Laurent power series: function in an annulus

THEOREM: (Laurent theorem)
Let f be a holomorphic function on an annulus (that is, a ring)

$$R = \{z \mid \alpha < |z| < \beta\}.$$

Then f can be expressed as a Laurent power series $f(z) = \sum_{i\in\mathbb{Z}} z^i a_i$ converging in R.

Proof: Same as Cauchy formula: for an annulus with components of the boundary denoted as ∂R_+ and ∂R_-, one has

$$\int_{\partial R_+} \frac{f(z)dz}{z-a} - \int_{\partial R_-} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}\, f(a),$$

This gives

$$2\pi\sqrt{-1}\, f(a) = \sum_{i\geqslant 0} a^i \int_{\partial R_+} f(z)(z^{-1})^{i+1} - \sum_{i\geqslant 0} a^{-i-1} \int_{\partial R_-} f(z) z^i$$

because $\frac{1}{z-a} = z^{-1}\sum_{i\geqslant 0}(az^{-1})^i$ for $|z| > |a|$ and $\frac{1}{z-a} = a^{-1}\sum_{i\geqslant 0}(a^{-1}z)^i$ for $|z| < |a|$. ■

REMARK: This theorem remains valid if $\alpha = 0$ and $\beta = \infty$.

Affine coordinates on $\mathbb{C}P^1$

DEFINITION: We identify $\mathbb{C}P^1$ with the set of pairs $x : y$ defined up to equivalence $x : y \sim \lambda x : \lambda y$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordimates**. **Affine coordinates** are $1 : z$ for $x \neq 0$, $z = y/x$ and $z : 1$ for $y \neq 0$, $z = x/y$. The corresponding gluing functions are given by the map $z \longrightarrow z^{-1}$.

DEFINITION: **Meromorphic function** is a quotient f/g, where f, g are holomorphic and $g \neq 0$.

REMARK: A holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}P^1$ is the same as a pair of maps $f : g$ up to equivalence $f : g \sim fh : gh$. **In other words, holomorphic maps $\mathbb{C} \longrightarrow \mathbb{C}P^1$ are identified with meromorphic functions on $\mathbb{C}$.**

REMARK: In homogeneous coordinates, an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ acts as $x : y \longrightarrow ax + by : cx + dy$. Therefore, in affine coordinates it acts as $z \longrightarrow \frac{az+b}{cz+d}$.

Möbius transforms

DEFINITION: **Möbius transform** is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: **The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.**

The following theorem will be proven next lecture.

THEOREM: **The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism**.