Teoria Ergódica Diferenciável

lecture 7: von Neumann ergodic theorem

Instituto Nacional de Matemática Pura e Aplicada

Misha Verbitsky, October 6, 2017

Hilbert spaces (reminder)

DEFINITION: Hilbert space is a complete, infinite-dimensional Hermitian space which is second countable (that is, has a countable dense set).

DEFINITION: Orthonormal basis in a Hilbert space H is a set of pairwise orthogonal vectors $\{x_\alpha\}$ which satisfy $|x_\alpha| = 1$, and such that H is the closure of the subspace generated by the set $\{x_\alpha\}$.

THEOREM: Any Hilbert space has a basis, and all such bases are countable.

Proof: A basis is found using Zorn lemma. If it's not countable, open balls with centers in x_α and radius $\varepsilon < 2^{-1/2}$ don't intersect, which means that the second countability axiom is not satisfied. ■

THEOREM: All Hilbert spaces are isometric.

Proof: Each Hilbert space has a countable orthonormal basis. ■

Real Hilbert spaces

DEFINITION: A Euclidean space is a vector space over $\mathbb{R}$ equipped with a positive definite scalar product g.

DEFINITION: **Real Hilbert space** is a complete, infinite-dimensional Euclidean space which is second countable (that is, has a countable dense set).

DEFINITION: **Orthonormal basis** in a Hilbert space H is a set of pairwise orthogonal vectors $\{x_\alpha\}$ which satisfy $|x_\alpha| = 1$, and such that H is the closure of the subspace generated by the set $\{x_\alpha\}$.

THEOREM: **Any real Hilbert space has a basis, and all such bases are countable.**

Proof: A basis is found using Zorn lemma. If it's not countable, open balls with centers in x_α and radius $\varepsilon < 2^{-1/2}$ don't intersect, which means that the second countability axiom is not satisfied. ■

THEOREM: **All real Hilbert spaces are isometric**.

Proof: Each Hilbert space has a countable orthonormal basis. ■

Adjoint maps

EXERCISE: Let (H, g) be a Hilbert space. Show that **the map $x \longrightarrow g(x, \cdot)$ defines an isomorphism $H \longrightarrow H^*$.**

DEFINITION: Let $A: H \longrightarrow H$ be a continuous linear endomorphism of a Hilbert space (H, g). Then $\lambda \longrightarrow \lambda(A(\cdot))$ map $A^*: H^* \longrightarrow H^*$. Identifying H and H^* as above, we interpret A^* as an endomorphism of H. It is called **adjoint endomorphism** (**Hermitian adjoint** in Hermitian Hilbert spaces).

REMARK: The map A^* satisfies $g(x, A(y)) = g(A^*(x), y)$. This relation is often taken as a definition of the adjoint map.

DEFINITION: An operator $U: H \longrightarrow H$ is **orthogonal** if $g(x, y) = g(U(x), U(y))$ for all $x, y \in H$.

CLAIM: An invertible operator U **is orthogonal if and only if $U^* = U^{-1}$.**

Proof: Indeed, orthogonality is equivalent to $g(x, y) = g(U^*U(x), y)$, which is equivalent to $U^*U = \mathrm{Id}$ because the form $g(z, \cdot)$ is non-zero for non-zero z. ■

Orthogonal maps and direct sum decompositions

LEMMA: Let $U: H \longrightarrow H$ be an invertible orthogonal map. Denote by H^U the kernel of $1-U$, that is, the space of U-invariant vectors, and let H_1 be the closure of the image of $1-U$. **Then $H = H^U \oplus H_1$ is an orthogonal direct sum decomposition.**

Proof: Let $x \in H^U$. Then

$$(U^* - 1)(x) = (U^* - 1)U(x) = (U^{-1} - 1)U(x) = (1-U)x = 0.$$

This gives $g(x, (U-1)y) = g((U^*-1)x, y) = 0$, hence $x \perp H_1$. Conversely, any vector x which is orthogonal to H_1 satisfies $0 = g(x, (U-1)y) = g((U^*-1)x, y)$, giving

$$0 = (U^* - 1)(x) = (U^* - 1)U(x) = (U^{-1} - 1)U(x) = (1-U)x.$$

■

Von Neumann erodic theorem

Corollary 1: Let $U: H \longrightarrow H$ be an invertible orthogonal map, and $U_n := \frac{1}{n}\sum_{i=0}^{n-1} U^i(x)$. **Then $\lim_n U_n(x) = P(x)$, for all $x \in H$ where P is orthogonal projection to H^U.**

Proof: By the previous lemma, it suffices to show that $\lim_n U_n = 0$ on H_1. However, the vectors of form $x = (1-U)(y)$ are dense in H_1, and for such x we have $U_n(x) = U_n(1-U)(y) = \frac{1-U^n}{n}(y)$, and it converges to 0 because $\|U^n\| = 1$. ∎

THEOREM: Let (M,μ) be a measure space and $T: M \longrightarrow M$ a map preserving the measure. Consider the space $L^2(M)$ of functions $f: M \longrightarrow \mathbb{R}$ with f^2 integrable, and let $T^*: L^2(M) \longrightarrow L^2(M)$ map f to T^*f. **Then the series $T_n(f) := \frac{1}{n}\sum_{i=0}^{n-1}(T^*)^i(f)$ converges in $L^2(M)$ to a T^*-invariant function.**

Proof: Corollary 1 implies that $T_n(f)$ converges to $P(f)$. ∎

The Hopf Argument

DEFINITION: Let M be a metric space with a Borel measure and $F: M \longrightarrow M$ a continuous map preserving measure. The **"stable foliation"** is an equivalence relation on M, with $x \sim y$ when $\lim_i d(F^n(x), F^n(y)) = 0$. The **"leaves"** of stable foliation are the equivalence classes.

THEOREM: (Hopf Argument) Any measurable, F-invariant function **is constant on the leaves of stable foliation** outside of a measure 0 set.

Proof: Let $A(f) := \lim_n \frac{1}{n}\sum_{i=0}^{n-1}(F^i)^* f$ be the map defined above. Since $A(f) = f$ for any F-invariant f, it suffices to prove that $A(f)$ is constant on leaves of the stable foliation only for $f \in \operatorname{im} A$. The Lipschitz L^2-integrable functions are dense in $L^1(M)$ by Stone-Weierstrass. Therefore it suffices to show that $A(f)$ is constant on leaves of the stable foliation when f is C-Lipschitz for some $C > 0$ and square integrable.

For any sequence $\alpha_i \in \mathbb{R}$ converging to 0, the sequence $\frac{1}{n}\sum_{i=0}^{n-1} \alpha_i$ also converges to 0. Therefore, whenever $x \sim y$, one has

$$A(f)(x) - A(f)(y) = \lim_n \sum_{i=0}^{n-1} f(F^i(x)) - f(F^i(y)) = 0$$

because $\alpha_i = |f(F^i(x)) - f(F^i(y))| \leqslant C d(F^i(x), F^i(y))$ converges to 0. ■

Stable and unstable foliations

DEFINITION: Let M be a metric space with a Borel measure and $F: M \longrightarrow M$ a homeomorphism preserving measure. The **"unstable foliation"** is a stable foliation for F^{-1}.

DEFINITION: The map F is called **pseudo-Anosov** if any leaf of stable foliation intersects any leaf of unstable foliation.

COROLLARY: A pseudo-Anosov map $F: M \longrightarrow M$ is always ergodic.

Proof: F is ergodic if all F-invariant $f \in L^2(M)$ are constant. However, al such f are constant on leaves of stable foliation and leaves on unstable foliation and these leaves intersect. ■

EXAMPLE: (Anosov diffeomorphism)
Let $A: T^2 \longrightarrow T^2$ be a linear map of a torus defined by $A \in SL(2, \mathbb{Z})$, with real eigenvalues $\alpha > 1$ and $\beta \in]0, 1[$, The eigenspace corresponding to β gives a stable foliation, the eigenspace corresponding to α the unstable foliation, hence **A is ergodic.**

Arnold's cat map

DEFINITION: The Arnold's cat map is $A: T^2 \longrightarrow T^2$ defined by $A \in SL(2, \mathbb{Z})$,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues of A are roots of $\det(t\,\mathrm{Id} - A) = (t-2)(t-1) - 1 = t^2 - 3t - 1$. This is a quadratic equation with roots $\alpha_\pm = \frac{3\pm\sqrt{5}}{2}$. On the vectors tangent to the eigenspace of α_-, the map A^n acts as $(\alpha_-)^n$, hence the stable foliation is tangent to these vectors. Similarly, unstable foliation is tangent to the eigenspace of α_+.