Teoria Ergódica Diferenciável

lecture 5: Weak-$*$ topology and Birkhoff Ergodic Theorem

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Weak-$*$ topology (reminder)

DEFINITION: Let M be a topological space, and $C^0_c(M)$ the space of continuous function with compact support. Any finite Borel measure μ defines a functional $C^0_c(M) \longrightarrow \mathbb{R}$ mapping f to $\int_M f\mu$. We say that a sequence $\{\mu_i\}$ of measures **converges in weak-$*$ topology** (or **in measure topology**) to μ if

$$\lim_i \int_M f\mu_i = \int_M f\mu$$

for all $f \in C^0_c(M)$. **The base of open sets of weak-$*$ topology** is given by $U_{f,]a,b[}$ where $]a,b[\subset \mathbb{R}$ is an interval, and $U_{f,]a,b[}$ is the set of all measures μ such that $a < \int_M f\mu < b$.

Tychonoff topology (reminder)

DEFINITION: Let $\{X_\alpha\}$ be a family of topological spaces, parametrized by $\alpha \in \mathcal{I}$. **Product topology**, or **Tychonoff topology** on the product $\prod_\alpha X_\alpha$ is topology where the open sets are generated by unions and finite intersections of $\pi_a^{-1}(U)$, where $\pi_a : \prod_\alpha X_\alpha$ is a projection to the X_a-component, and $U \subset X_a$ is an open set.

REMARK: Tychonoff topology is also called **topology of pointwise convergence**, because the points of $\prod_\alpha X_\alpha$ can be considered as maps from the set of indices $\mathcal{I}$ to the corresponding X_α, and a sequence of such maps converges if and only if it converges for each $\alpha \in \mathcal{I}$.

REMARK: Consider a finite measure as an element in the product of $C_c^0(M)$ copies of $\mathbb{R}$, that is, as a continuous map from $C_c^0(M)$ to $\mathbb{R}$. **Then the weak-$*$ topology is induced by the Tychonoff topology on this product.**

Radon measures

DEFINITION: Radon measure (or **regular measure** on a locally compact topological space M is a Borel measure μ which satisfies the following assumptions.

1. μ is finite on all compact sets.

2. For any Borel set E, one has $\mu(E) = \inf \mu(U)$, where infimum is taken over all open U containing E.

3. For any open set E, one has $\mu(E) = \sup \mu(K)$, where infimum is taken over all compact K contained in E.

DEFINITION: Uniform topology on functions is induced by the metric $d(f, g) = \sup |f - g|$.

Riesz representation theorem

Riesz representation theorem: Let M be a metrizable, locally compact topological space, and $C_c^0(M)^*$ the space of functionals continuous in uniform topology. **Then Radon can be characterized as functionals $\mu \in C_c^0(M)^*$ which are non-negative on all non-negative functions.**

Proof: Clearly, all measures give such functionals. Conversely, consider a functional $\mu \in C_c^0(M)^*$ which is non-negative on all non-negative functions. Given a closed set $K \subset M$, the characteristic function χ_K can be obtained as a monotonously decreasing limit of continuous functions f_i which are equal to 1 on K **(prove it).** Define $\mu(K) := \lim_i \mu(f_i)$; this limit is well defined because the sequence $\mu(f_i)$ is monotonous. This gives an additive Borel measure on M **(prove it).** ■

Space of measures and Tychonoff topology (reminder)

REMARK: (**Tychonoff theorem**)
A product of any number of compact spaces is compact.

This theorem is hard and its proof is notoriously counter-intiutive. However, from Tychonoff the following theorem follows immediately.

THEOREM: Let M be a compact topological space, and $\mathcal{P}$ the space of probability measures on M equipped with the measure topology. **Then $\mathcal{P}$ is compact.**

Proof. Step 1: For any probability measure on M, and any $f \in C^0_c(M)$, one has $\min(f) \leqslant \int_M f\mu \leqslant \max(f)$. Therefore, μ can be considered as an element of the product $\prod_{f\in C^0_c(M)}[\min(f), \max(f)]$ of closed intervals indexed by $f \in C^0_c(M)$, and **Tychonoff topology on this product induces the weak-$*$ topology.**

Step 2: A closed subset of a compact set is again compact, hence **it suffices to show that all limit points of $\mathcal{P} \subset \prod_{f\in C^0_c(M)}[\min(f), \max(f)]$ are probability measures.** This is implied by Riesz representation theorem. The limit measure satisfies $\mu(M) = 1$ because the constant function $f = 1$ has compact support, hence $\lim \int_M \mu_i = \int_M \mu$ whenever $\lim_i \mu_i = \mu$. ■

The space of Lipschitz functions is second countable

DEFINITION: An ε**-net** in a metric space M is a subset $Z \subset M$ such that any $m \in M$ lies in an ε-ball with center in Z.

REMARK: A metric space is compact **if and only if it has a finite ε-net for each $\varepsilon > 0$ (prove it)**.

Claim 1: Let M be a compact metrizable topological space. **Then the space of C-Lipschitz functions has a countable dense subset.**

Proof. Step 1: Let Z be a finite ε/C-net in M_0. Then for any C-Lipschitz functions f, g, one has

$$\left| \sup_{m\in M} |f-g| - \sup_{z\in Z} |f-g| \right| < 2\varepsilon,$$

because for each $m \in M$ there exists $m' \in Z$ such that $d(m, m') < \varepsilon/C$, and then $|f(m) - f(m')| < C\varepsilon/C = \varepsilon$, giving $|f(m) - g(m)| < |f(m') - g(m')| + 2\varepsilon$.

The space of Lipschitz functions is second countable (2)

Proof. Step 1: Let Z be a finite ε/C-net in M_0. Then for any C-Lipschitz functions f, g,

$$\left|\sup_{m\in M} |f-g| - \sup_{z\in Z} |f-g|\right| < 2\varepsilon.$$

Step 2: Let R_ε be the set of all functions on Z with values in $\mathbb{Q}$. For each $\varphi \in R_\varepsilon$ denote by U_φ an open set of all C-Lipschitz functions f satisfying $\max_{z\in Z} |f(z)-\varphi(z)| < \varepsilon$. Then for all $f, g \in U_\varphi$, one has $\max_{z\in Z} |f(z)-g(z)| < 2\varepsilon$, and by Step 1 this gives $\sup_{m\in M} |f-g| < 4\varepsilon$.

Step 3: The set of all such U_φ is countable; choosing a function f_φ in each non-empty U_φ, we use $\sup_{m\in M} |f-g| < 4\varepsilon$ to see that $\{f_\varphi\}$ is a countable 4ε-net in the space of C-Lipschitz functions. ■

COROLLARY: Let M be a compact metrizable topological space. **Then $C^0_c(M)$ has a countable dense subset.**

Proof: Using Claim 1, we see that it is sufficient to show that Lipschitz functions are dense in the set of all continuous functions; this follows from the Stone-Weierstrass theorem. ■

Tychonoff theorem for countable families

REMARK: Let $\{F_i\}$ be a countable, dense set in $C^0(M)$. Then any measure μ is determined by $\int_M F_i\mu$, and **weak-$*$ topology is topology of pointwise convergence on F_i.** This implies that **compactness of the space of measures is implied by the compactness of the product $\prod_{F_i}[\min(F_i), \max(F_i)]$, which is countable.**

THEOREM: (Countable Tychonoff theorem)
A countable product of metrizable compacts is compact.

Proof: Let $\{M_i\}$ be a countable family of metrizable compacts. We need to show that the space of sequences $\{a_i \in M_i\}$ with topology of pointwise convergence is compact. Take a sequence $\{a_i(j)\}$ of such sequences, and replace it by a subsequence $\{a'_i(j) \in M_i\}$ where $a_1(i)$ converges. Let $b_1 := \lim a'_i(1)$. Replace this sequence by a subsequence $\{a''_i(j) \in M_i\}$ where $a_2(i)$ converges. Put $b_2 = \lim_i a''_i(2)$ and so on. Then $\{b_i\}$ is a limit point of our original sequence $\{a_i(j)\}$. By Heine-Borel, compactness for second countable spaces is equivalent to sequential compactness, hence $\prod_i M_i$ is compact. ∎

Fréchet spaces

DEFINITION: A seminorm on a vector space V is a function $\nu : V \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying

1. $\nu(\lambda x) = |\lambda|\nu(x)$ for each $\lambda \in \mathbb{R}$ and all $x \in V$

2. $\nu(x+y) \leqslant \nu(x) + \nu(y)$.

DEFINITION: We say that **topology on a vector space V is defined by a family of seminorms** $\{\nu_\alpha\}$ if the base of this topology is given by the finite intersections of the sets

$$B_{\nu_\alpha,\varepsilon}(x) := \{y \in V \quad | \quad \nu_\alpha(x-y) < \varepsilon\}$$

("open balls with respect to the seminorm"). It is **complete** if each sequence $x_i \in V$ which is Cauchy with respect to each of the seminorms converges.

DEFINITION: A **Fréchet space** is a Hausdorff second countable topological vector space V with the topology defined by a countable family of seminorms, complete with respect to this family of seminorms.

Seminorms and weak-$*$ topology

REMARK: Let M be a manifold and W be the subspace in functionals on $C^0_c(M)$ generated by all Borel measures ("the space of signed measures"). Recall that **the Hahn decomposition** is a decomposition of $\mu \in W$ as $\mu = \mu_+ - \mu_-$, where μ_+, μ_- are measures with non-intersecting support.

EXAMPLE: Then the weak-$*$ topology is defined by a countable family of seminorms. Indeed, we can choose a dense, countable family of functions $f_i \in C^0_c(M)$, and define the seminorms ν_{f_i} on measures by $\nu_{f_i}(\mu) := \int_M f_i\mu$ extending it to W by $\nu_{f_i}(\mu) = \int_M f_i\mu_+ + \int_M f_i\mu_-$, where $\mu = \mu_+ - \mu_-$ is the Hahn decomposition.

EXERCISE: Prove that **the space W of signed measures with weak-$*$ topology is complete.**

REMARK: This exercise is hard, but **for our purposes it is sufficient to replace W by its seminorm completion $\overline{W}$.** Since the space of finite measures is compact, it is also complete in $\overline{W}$.

Fréchet spaces: some examples

EXERCISE: Prove that the **space of continuous functions on $\mathbb{R}^n$ with topology of uniform convergence with compact support is Fréchet.**

EXERCISE: Prove that the space of smooth functions on a compact manifold **admits a structure of a Fréchet space**.

Existence of invariant measures

Further on, we shall prove the following theorem

Theorem 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A : V \longrightarrow V$ a continuous linear map which preserves K. **Then there exists a point $z \in K$ such that $A(z) = z$.**

We shall prove this theorem in the next slide.

COROLLARY: Let M be a compact topological space and $f : M \longrightarrow M$ a continuous map. **Then there exists an f-invariant probability measure on M.**

Proof: Take the compact space $K \subset W$ of all probability measures, and let $A : K \longrightarrow K$ map μ to $f_*\mu$. Then A has a fixed point, as follows from Theorem 1. ■

Linear maps on convex compact sets

Theorem 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A: V \longrightarrow V$ a continuous linear map which preserves K. **Then there exists a point $z \in K$ such that $A(z) = z$.**

Proof: Consider the linear map $A_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} A^i(x)$. Since it is an average of points in K, one has $A_n(x) \in K$. Let $z \in K$ be a limit point of the sequence $\{A_n(x)\}$ for some $x \in K$. Since

$$(1-A)A_n(x) = \frac{(1-A)\left(\sum_{i=0}^{n-1} A^n\right)}{n} = \frac{1-A^n}{n},$$

for each seminorm ν_i on V one has

$$\nu(A(A_n(x)) - A_n(x)) < \frac{C}{n},$$

where

$$C := \sup_{x,y \in K} \nu(x-y).$$

By continuity of ν, this gives $\nu(A(z) - z) < \frac{C}{n}$ for each $n > 0$, hence $A(z) = z$.

■

Linear maps on convex compact sets: properties of the limit

Claim 1: Let $K \subset V$ be a compact, convex subset of a topological vector space with topology defined by a family of seminorms, and $A: V \longrightarrow V$ a continuous linear map which preserves K. Consider the map $A_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} A^n(x)$, and let $\operatorname{Map}(K, K)$ be the space of maps from K to itself with the Tychonoff topology. Then $\{A_n\}$ has a subsequence converging to a linear map B from K to itself. Consider B as a linear map from the space $V' \subset V$ generated by K to itself. for two such limits B_1 and B_2, the difference $E := B_1 - B_2$ satisfies $\operatorname{im} E \subset V_0, \ker E \subset V_0$, where $V_0 = \ker(1 - A) \cap V'$.

Proof. Step 1: Consider the space $\operatorname{Map}(K, K)$ of maps from K to itself with the product topology. By Tychonoff theorem, it is compact. The set of linear maps is closed in $\operatorname{Map}(K, K)$ **(prove it)**. Then **the sequence $\{A_n \in \operatorname{Map}(K, K)\}$ has a limit point $B: K \longrightarrow K$** which is a linear map on K. Then **B defines a linear (possibly discontinuous) endomorphism of V'.**

Step 2: Since $(1 - A)A_n(x) = \frac{1-A^n}{n}$, one has $(1 - A)B = B(1 - A) = 0$. This implies that $\operatorname{im} B \subset V_0$. Since $B\big|_{V_0} = A$, we also have $E\big|_{V_0} = V_0$. ■

Birkhoff Ergodic Theorem

Lemma 1: Let $C>0$ be a constant, ν a measure on S, and $K_{C,\nu}$ be the space of measures μ on S which satisfy $\mu(U)\leqslant C\nu(U)$. for all measurable sets. **Then K_ν is closed in weak-$*$ topology. Proof:** $K_{C,\nu}=\bigcap_{f\in C^0_c(M)} K_f$, where $K_f=\{\text{measures } \mu \mid \int_S |f|\mu \leqslant C\int_S |f|\nu.\}$ ■

THEOREM: (Birkhoff Ergodic Theorem) Let $f: M\longrightarrow M$ be a continuous map on a compact topological space, and μ a probability measure. Assume that $\mu=\Phi\nu$, where $f_*\nu=\nu$, and $|\Phi|<C$ is a bounded measurable function. **Then the sequence $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1}(f_*)^i\mu$ converges to a probability measure.**

Proof. Step 1: The sequence $\mu_n := \frac{1}{n}\sum_{i=0}^{n-1}(f_*)^i\mu$ has a limit point μ' which is absolutely continuous with respect to ν by Lemma 1. Moreover, the function $\Psi := \frac{\mu'}{\nu}$ is bounded by the same constant C. Since $|\mu_n - f_*\mu_n| < \frac{|\mu_n|-|f_*^n\mu_n|}{n}$, **the limit function Ψ is f-invariant.**

Step 2: Consider the map $E: K\longrightarrow V_0$ of Claim 1. Restricted to functions which we consider as signed measures, this map defines an f_*-invariant V_0-valued functional $\tilde\nu: C^0_c(M)\longrightarrow V_0$ which is absolutely continuous with respect to μ. Composing it with a linear functional, and applying Radon-Nikodym, we obtain an integrable f_*-invariant function $\Phi\in L^1(M)$. Then $\nu(\Phi)\neq 0$, because $\int_M \Phi^2>0$. This is impossible, because $E|_{V_0}=0$. ■