Teoria Ergódica Diferenciável

lecture 3: Examples of ergodic measures

Instituto Nacional de Matemática Pura e Aplicada

Misha Verbitsky, August 16, 2017

σ-algebras and measures (reminder)

DEFINITION: Let M be a set **A σ-algebra** of subsets of X is a Boolean algebra $\mathfrak{A} \subset 2^X$ such that for any countable family $A_1, ..., A_n, ... \in \mathfrak{A}$ the union $\bigcup_{i=1}^{\infty} A_i$ is also an element of $\mathfrak{A}$.

REMARK: We define the operation of addition on the set $\mathbb{R} \cup \{\infty\}$ in such a way that $x + \infty = \infty$ and $\infty + \infty = \infty$. On finite numbers the addition is defined as usually.

DEFINITION: A function $\mu : \mathfrak{A} \longrightarrow \mathbb{R} \cup \{\infty\}$ is called **finitely additive** if for all non-intersecting $A, B \in \mathfrak{U}$, $\mu(A \coprod B) = \mu(A) + \mu(B)$. The sign $\coprod$ denotes union of non-intersecting sets. μ is called **σ-additive** if $\mu(\coprod_{i=1}^{\infty} A_i) = \sum \mu(A_i)$ for any pairwise disjoint countable family of subsets $A_i \in \mathfrak{A}$.

DEFINITION: **A measure** in a σ-algebra $\mathfrak{A} \subset 2^X$ is a σ-additive function $\mu : \mathfrak{A} \longrightarrow \mathbb{R} \cup \{\infty\}$.

EXAMPLE: Let X be a topological space. The **Borel σ-algebra** is a smallest σ-algebra $\mathfrak{A} \subset 2^X$ containing all open subsets. **Borel measure** is a measure on Borel σ-algebra.

Measurable maps and measurable functions (reminder)

DEFINITION: Let X, Y be sets equipped with σ-algebras $\mathfrak{A} \subset 2^X$ and $\mathfrak{B} \subset 2^Y$. We say that a map $f: X \longrightarrow Y$ is **compatible with the σ-algebra**, or **measurable**, if $f^{-1}(B) \in \mathfrak{A}$ for all $B \in \mathfrak{B}$.

REMARK: This is similar to the definition of continuity. In fact, **any continuous map of topological spaces is compatible with Borel σ-algebras.**

DEFINITION: Let X be a space with σ-algebra $\mathfrak{A} \subset 2^X$. A function $f: X \longrightarrow \mathbb{R}$ is called **measurable** if f is compatible with the Borel σ-algebra on $\mathbb{R}$, that is, if the preimage of any Borel set $A \subset \mathbb{R}$ belongs to $\mathfrak{A}$.

DEFINITION: Let X, Y be sets equipped with σ-algebras $\mathfrak{A} \subset 2^X$ and $\mathfrak{B} \subset 2^Y$, $f: X \longrightarrow Y$ a measurable map. Let μ be a measure on X. Consider the function $f_*\mu$ mapping $B \in \mathfrak{B}$ to $\mu(f^{-1}(B))$.

EXERCISE: **Prove that $f_*\mu$ is a measure on Y.**

DEFINITION: The measure $f_*\mu$ is called **the pushforward measure**, or **pushforward** of μ.

Ergodic measures (reminder)

REMARK: Let M, μ be a space with measure. We say that **"property P holds for almost all $x \in M$"** when property P holds for all $x \in M$ outside of a measure 0 subset.

DEFINITION: Let Γ be a group acting on a measured space $(M.\mu)$ and preserving its σ-algebra. We say that the Γ-action is **ergodic** if for each Γ-invariant, measurable set $U \subset M$, either $\mu(U) = 0$ or $\mu(M\backslash U) = 0$. In this case μ is called **an ergodic measure**.

THEOREM: Let M be a second countable topological space, and μ a Borel measure on M. Let Γ be a group acting on M by homeomorphisms. Suppose that any non-empty open subset of M has positive measure, and action of Γ is ergodic. Then **for almost all $x \in M$, the orbit $\Gamma \cdot x$ is dense in M.**

THEOREM: Let (M, μ) be a space with finite measure, and Γ a group acting on M and preserving the measure. Then the following are equivalent.

(a) The action of Γ is ergodic.

(b) For each integrable, Γ-invariant function $f : M \longrightarrow \mathbb{R}$, f is constant almost everywhere.

Hilbert spaces

DEFINITION: **Hilbert space** is a complete, infinite-dimensional Hermitian space which is second countable (that is, has a countable dense set).

DEFINITION: **Orthonormal basis** in a Hilbert space H is a set of pairwise orthogonal vectors $\{x_\alpha\}$ which satisfy $|x_\alpha| = 1$, and such that H is the closure of the subspace generated by the set $\{x_\alpha\}$.

THEOREM: **Any Hilbert space has a basis, and all such bases are countable.**

Proof: A basis is found using Zorn lemma. If it's not countable, open balls with centers in x_α and radius $\varepsilon < 2^{-1/2}$ don't intersect, which means that the second countability axiom is not satisfied. ■

THEOREM: **All Hilbert spaces are isometric**.

Proof: Each Hilbert space has a countable orthonormal basis. ■

Fourier series

EXAMPLE: Let (M, μ) be a space with measure. Consider the space V of measurable functions $f: M \longrightarrow \mathbb{C}$ such that $\int_M |f|^2 \mu < \infty$. For each $f, g \in V$, the integral $\int f \bar{g} \mu$ is well defined, by Cauchy inequality: $\int |fg| \mu < \sqrt{\int_M |f|^2 \mu \int_M |g|^2 \mu}$. This gives a Hermitian form on V Let $L^2(M)$ denote the completion of V with respect to this metric. It is called **the space of square-integrable functions on M**. Its elements are called **L^2-functions.**

CLAIM: ("Fourier series") Functions $e_k(t) = e^{2\pi\sqrt{-1}\, kt}$, $k \in \mathbb{Z}$ on $S^1 = \mathbb{R}/\mathbb{Z}$ **form an orthonormal basis in the space $L^2(S^1)$**.

Proof: Orthogonality is clear from $\int_{S^1} e^{2\pi\sqrt{-1}\, kt} dt = 0$ for all $k \neq 0$ (prove it).

To show that the space of Fourier polynomials $\sum_{i=-n}^{n} a_k e_k(t)$ is dense in the space of continuous functions on circle, use the Stone-Weierstrass approximation theorem. ■

Integrable functions on spaces with finite measure are square integrable

LEMMA: Let (M, μ) be a space with finite measure (that is, $\int_M \mu < \infty$). **Then any square integrable function is integrable.**

Proof: Cauchy inequality gives

$$\int |f|\mu < \sqrt{\int_M |f|^2\mu \int_M 1\mu}.$$

■

Hilbert spaces and ergodicity

COROLLARY: Let (M, μ) be a space with finite measure and Γ a group acting on M. Then the following are equivalent.

(a) The action of Γ is ergodic.

(b) For each square integrable, Γ-invariant function $f: M \longrightarrow \mathbb{R}$, f is constant almost everywhere.

(c) For each integrable, Γ-invariant function $f: M \longrightarrow \mathbb{R}$, f is constant almost everywhere.

Proof: (c) implies (b) by the previous lemma, (b) implies (a) because a characteristic function of a measurable subset is square integrable, and (a) $\Rightarrow$ (c) was already proven. ■

COROLLARY: Let α be an irrational number, and $\varphi_\alpha: S^1 \longrightarrow S^1$ be a rotation by $\pi\alpha$. **Then $\pi\alpha$ is ergodic.**

Proof: Let $e_k(t) = e^{2\pi\sqrt{-1}\,kt}$ be the Fourier series basis. For any L^2-function $f = \sum_{k\in\mathbb{Z}} a_k e_k$, one has $\varphi_\alpha^*(f) = \sum_{k\in\mathbb{Z}} e^{\sqrt{-1}\,k\pi\alpha} a_k e_k$. Since α is irrational, $e^{\sqrt{-1}\,k\pi\alpha} \neq 1$ for all $k \neq 0$, and **the action of φ_α on $L^2(S_1)$ has no non-constant invariant L^2-functions.** ■

Ergodic systems

DEFINITION: Let M be a space with measure, and $\Phi : M \longrightarrow M$ a measurable map. We say that a funcion $f : M \longrightarrow \mathbb{R}$ is **Φ-invariant** if $\Phi^*(f) = f$ almost everywhere, that is, $\Phi \circ f = f$. We say that a subset $U \subset M$ **is Φ-invariant**, if $\Phi^{-1}(U) = U$ up to measure 0 subset.

REMARK: A subset $U \subset M$ is Φ-invariant **if and only if the corresponding characteristic function χ_U is Φ-invariant.**

DEFINITION: **Dynamical system** is a triple (M, μ, Φ), where (M, μ) is a space with measure, and $\Phi : M \longrightarrow M$ a measurable map.

DEFINITION: Let (M, μ, Φ) be a dynamical system. It is called **ergodic system** if any Φ-invariant measurable subset has full measure or has zero measure.

The doubling map

THEOREM: Let (M, μ, Φ) be a dynamical system, with μ finite. Then the following are equivalent.

(a) (M, μ, Φ) **is ergodic.**

(b) For each square integrable, Φ-invariant function $f: M \longrightarrow \mathbb{R}$, f is constant almost everywhere.

(c) For each integrable, Φ-invariant function $f: M \longrightarrow \mathbb{R}$, f is constant almost everywhere.

Proof: Same as above. ■

EXAMPLE: Let $S^1 = \mathbb{R}/\mathbb{Z}$, and **the doubling map** $\mathfrak{D}$ map α to 2α.

CLAIM: The doubling map defines an ergodic system on S^1.

Proof: Let $e_k(t) = e^{2\pi\sqrt{-1}\,kt}$ be the Fourier series basis. For any L^2-function $f = \sum_{k\in\mathbb{Z}} a_k e_k$, one has $\mathfrak{D}^*(f) = \sum_{k\in\mathbb{Z}} a_k e_{2k}$. Then the action of φ_α on $L^2(S_1)$ has no non-constant invariant L^2-functions. ■

Lebesque measure (reminder)

DEFINITION: Pseudometric on X is a function $d: X \times X \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric and satisfies the triangle inequality and $d(x,x)=0$ for all $x \in X$. In other words, pseudometric is a metric which can take 0 on distinct points.

EXERCISE: Let $\mathfrak{A} \subset 2^X$ be a Boolean algebra with positive, additive function μ. Given $U, V \in 2^X$, denote by $U \triangle V$ their **symmetric difference**, that is, $U \triangle V := (U \cup V) \backslash (U \cap V)$. **Prove that the function $d_\mu(U,V) := \mu(U \triangle V)$ defines a pseudometric on $\mathfrak{A}$.**

DEFINITION: Let $\mathfrak{A} \subset 2^X$ be a Boolean algebra with positive, additive function μ. A set $U \subset X$ **has measure 0** if for each $\varepsilon > 0$, U can be covered by a union of $A_i \in \mathfrak{A}$, that is, $U \subset \bigcup_{i=1}^\infty A_i$, with $\sum_{i=0}^\infty \mu(A_i) < \varepsilon$.

REMARK: Consider a completion of $\mathfrak{A}$ with respect to the pseudometric d_μ. A limit of a Cauchy sequence $\{A_i\} \subset \mathfrak{A}$ can be realized as an element of 2^X; this realization is unique up to a set of measure 0. A set which can be obtained this way is called **a Lebesgue measurable set**. Extending μ to the metric completion of $\mathfrak{A}$ by continuity, we obtain **the Lebesgue measure** on the σ-algebra of Lebesgue measurable sets.

REMARK: This construction is also used **for constructing Borel measures.**

Lebesque approximation theorem

THEOREM: (Lebesque approximation theorem)
Let M be a topological space, and $\mathfrak{A}_0$ a Boolean algebra of Borel subsets such that the corresponding σ-algebra $\mathfrak{A}$ contains all Borel subsets. Consider an additive, finite measure on $\mathfrak{A}_0$ which continually extends to its completion $\mathfrak{A}$. **Then for each $X \in \mathfrak{A}$ and each $\varepsilon > 0$ there exists $X_0 \in \mathfrak{A}_0$ such that $\mu(X \triangle X_0) < \varepsilon$.**

Proof: X is a limit of a Cauchy sequence from $\mathfrak{A}_0$. ■

EXERCISE: Let C be a cube in $\mathbb{R}^n$, and $\mathfrak{A}_0$ be an algebra generated by parallelepipeds. **Prove that for each each $T \in \mathfrak{A}_0$, there exists an open subset $T' \in \mathfrak{A}_0$ such that $\mu(T \triangle T') = 0$.**

COROLLARY: For each measurable subset $X \subset C$, and each $\varepsilon > 0$, **there exists an open subset $X_0 \in \mathfrak{A}_0$ such that $\mu(X \triangle X_0) < \varepsilon$.** ■

Lebesque approximation theorem for Bernoulli space

DEFINITION: Let P be a finite set, $P^{\mathbb{Z}}$ the product of $\mathbb{Z}$ copies of P, $\Sigma \subset \mathbb{Z}$ a finite subset, and $\pi_\Sigma : P^{\mathbb{Z}} \longrightarrow P^{|\Sigma|}$ projection to the corresponding components. **Tychonoff topology**, or **product topology** is topology where the base of open sets are given by **cylindrical sets** $C_R := \pi_\Sigma^{-1}(R)$, where $R \subset P^{|\Sigma|}$ is any subset.

REMARK: For Bernoulli space, **a complement to an cylindrical set is again an open set,** and the cylindrical sets **form a Boolean algebra.**

DEFINITION: Bernoulli measure on $P^{\mathbb{Z}}$ is a measure μ such that $\mu(C_R) := \frac{|R|}{|P|^{|\Sigma|}}$.

THEOREM: (Lebesque approximation theorem)
For each Lebesgue measurable set $S \subset P^{\mathbb{Z}}$ and $\varepsilon > 0$, there exists a cylindrical subset $C_R = \pi_\Sigma^{-1}(R)$ **such that** $\mu(C_R \triangle X) < \varepsilon$**.**

Proof: The σ-algebra of Lebesgue measurable sets is by definition a completion of the Boolean algebra of cylindrical sets. ■

Bernoulli shifts are ergodic

We represent an element of Bernulli space $P^{\mathbb{Z}}$ by a sequence $a_{-n}, a_{-n+1}, ..., a_0, a_1, ...,$ with $a_i \in P$.

DEFINITION: Bernoulli shift maps a sequence $a_{-n}, a_{-n+1}, ..., a_0, a_1, ...$ to the sequence $b_{-n}, b_{-n+1}, ..., b_0, b_1, ...,$ $b_i = a_{i-1}$.

CLAIM: The corresponding $\mathbb{Z}$-action is ergodic on the Bernoulli space.

Proof. Step 1: Let $C_R = \pi_{\Sigma}^{-1}(R)$ and $C_{R'} = \pi_{\Sigma'}^{-1}(R')$ be two open sets, where $\Sigma \subset \mathbb{Z}$ and $\Sigma' \subset \mathbb{Z}$ don't intersect. Then $\mu(C_R \cap C_{R'}) = \mu(C_R)\mu(C_{R'})$. Indeed,

$$\mu(C_R \cap C_{R'}) = \frac{|R||R'|}{|P|^{|\Sigma|+|\Sigma'|}}.$$

This gives $\mu(C_R \triangle C_{R'}) = \mu(C_R) + \mu(C_{R'}) - \mu(C_R)\mu(C_{R'})$.

Bernoulli shifts are ergodic (2)

CLAIM: The Bernoulli shift action is ergodic on the Bernoulli space.

Proof. Step 1: Let C_R and $C_{R'}$ be two cylindrical sets, where $\Sigma \subset \mathbb{Z}$ and $\Sigma' \subset \mathbb{Z}$ don't intersect. **Then** $\mu(C_R \triangle C_{R'}) = \mu(C_R) + \mu(C_{R'}) - 2\mu(C_R)\mu(C_{R'})$**.**

Step 2: Let $U \subset P^{\mathbb{Z}}$ be a shift-invariant subset with $\varepsilon < \mu(U) < 1-\varepsilon$, and $C_R \subset P^{\mathbb{Z}}$ an open subset satisfying $\mu(C_R \triangle U) < \delta$, for a given $\delta < \frac{1}{4}\varepsilon$. Such C_R exists by Lebesgue approximation theorem. Replacing U by its complement if necessary, we may assume that $\varepsilon < \mu(U) \leqslant 1/2$, giving $\varepsilon - \delta < \mu(C_R) < 1/2 + \delta$.

Step 3: Denote by Φ a sufficiently big power of the Bernoulli shift such that $\mu(C_R \cap \Phi(C_R)) = \mu(C_R)^2$ (Step 1). Then

$$\mu(C_R \triangle \Phi(C_R)) = 2\mu(C_R) - 2\mu(C_R)^2 = 2\mu(C_R)(1 - \mu(C_R)).$$

Since $\varepsilon - \delta < \mu(C_R) < 1/2 + \delta$, this gives

$$\mu(C_R \triangle \Phi(C_R)) = 2\mu(C_R) - 2\mu(C_R)^2 > 2(\varepsilon - \delta)(1/2 - \delta) \geqslant \frac{7}{16}\varepsilon.$$

(for the last inequality use $\delta < \frac{1}{8}\varepsilon$). Since U is Φ-invariant, we have

$$\frac{7}{16}\varepsilon < \mu(C_R \triangle \Phi(C_R)) \leqslant \mu(C_R \triangle U) + \mu(\Phi(C_R) \triangle \Phi(U)) < 2\delta < \frac{1}{4}\varepsilon,$$

giving a contradiction. ■

Rotation and doubling are ergodic via Lebesgue approximation

EXERCISE: Prove that irrational rotations of a circle are ergodic using the Lebesgue approximation theorem.

EXERCISE: Prove that the doubling map is ergodic using the Lebesgue approximation theorem.