Teoria Ergódica Diferenciável: final exam

Rules: Every student receives from me a list of 10 exercises (chosen randomly), and has to solve as many of them as you can by due date. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solutions, using your notes. Please learn proofs of all results you will be using on the way (you may put them in your notes). Please contact me by email `verbit2000[]gmail.com` when you are ready.

The final score N is obtained by summing up the points from the exam problems and the course assignments.[1]

Marks: C when $40 \leqslant N < 70$, B when $70 \leqslant N < 100$, A when $100 \leqslant N < 130$, A+ when $N \geqslant 130$.

1 Cesàro summation and ergodicity

Exercise 1.1. Let (M, μ) be a space with finite measure, and $T: M \longrightarrow M$ a measure-preserving bijective map. Prove that for any measurable function $f: M \longrightarrow \mathbb{R}$,

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^i(f) = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} T^{-i}(f)$$

almost everywhere.

Definition 1.1. A sequence $\{x_i\}$ in a measured space (M, μ) is called **equidistributed** if the sequence $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{x_i}$ converges to μ.

Exercise 1.2 (20 points). Prove that the sequence $a_n = \log(n) \mod \mathbb{Z}$ is not equidistributed in $S^1 = \mathbb{R}/\mathbb{Z}$.

Exercise 1.3. Consider the map $T(x) = 1 - |2x - 1|$ from the interval $[0, 1]$ to itself, and let μ be the Lebesgue measure. Prove that μ is T-invariant.

a. (20 points) Prove or disprove that $([0, 1], \mu, T)$ is ergodic.

b. (10 points) Prove or disprove that $([0, 1], \mu, T)$ is uniquely ergodic.

Exercise 1.4 (20 points). Let μ be an ergodic probabilistic measure on (M, T), where M is compact and T continuous and bijective. Prove that there exists $x \in M$ such that $\mu = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)}$ where δ_x denotes the Dirac (atomic) measure in x.

Definition 1.2. Let $T: M \longrightarrow M$ be a surjective measurable map preserving the measure μ. **The extension** $\hat{M}$ is the set of sequences $x_0, x_{-1}, x_{-2}, ...$, such that $T(x_{-i}) = x_{-i+1}$. Denote by $\Pi_0: \hat{M} \longrightarrow M$ the forgetful map mapping a sequence $\{x_i\}$ to x_0. We define a measure $\hat{\mu}$ on $\hat{M}$ by $\hat{\mu}(U) := \mu(\Pi_0(U))$. Let $\hat{T}(\{x_0, x_{-1}, x_{-2}, ..., \}) = \{T(x_0), x_0, x_{-1}, x_{-2}, ...\}$

Exercise 1.5 (20 points). In these assumptions, prove that $\hat{T}: \hat{M} \longrightarrow \hat{M}$ is measure-preserving, and, moreover, T is ergodic if and only if $\hat{T}$ is ergodic.

[1] `http://www.verbit.ru/IMPA/Ergodic-2017/res-scores.txt`

Definition 1.3. Let $f: M \longrightarrow M$ be a bijective map. Define **suspension** $\Sigma_f(M)$ as a quotient $\mathbb{R} \times M$ by the equivalence $(t, m) \sim (t+1, f(m))$. The suspension is equipped with a natural map $\Sigma_f(M) \longrightarrow \mathbb{R}/\mathbb{Z}$ to the quotient of $\mathbb{R}$ by the equivalence relation $t \sim t+1$, identified with the circle S^1. If M has a measure μ and f is measure-preserving, define the measure $\tilde{\mu}$ on $\Sigma_f(M)$ by taking the standard measure ρ on a circle and putting $\mu(B \times I) = \mu(B_X)\rho(I)$ for any interval $I \subset [x, x+1]$ and any Borel set $B \subset X$. Define the flow $F_t: \mathbb{R} \times \Sigma_f(M) \longrightarrow \Sigma_f(M)$ by $F_r(t, m) = (t+r, m)$. This map is called **suspension of** f.

Exercise 1.6 (20 points). Prove that (M, μ, T) is ergodic if and only if F_t defines an ergodic action of $\mathbb{R}$ on $(\Sigma_f(M), \tilde{\mu})$.

Exercise 1.7 (20 points). Let T^n be a torus, $T^n = \mathbb{R}^n/\mathbb{Z}^n$, and $A \in SL(\mathbb{Z}^n)$ an invertible matrix with integer coefficients. Assume that A acts on T^n with dense orbits. Prove that this action is ergodic, or find a counterexample.

Exercise 1.8 (20 points). Let $T: M \longrightarrow M$ be an isometry of a compact metric space which has a dense orbit, and Vol the Riemannian volume measure. Prove that (M, Vol, T) is ergodic.

Exercise 1.9 (20 points). Let (M, μ, T) be an ergodic dynamical system. Prove that (M, μ, T^k) is ergodic for all $k > 0$, or find a counterexample.

2 Spectral theory

Exercise 2.1 (20 points). Let $O(H)$ be the group of linear isometries of a Hilbert space, and $\phi: \mathbb{R} \longrightarrow O(H)$ a continuous group homomorphism mapping t to U_t. Let $H_0 := \bigcap_{t\in\mathbb{R}} \mathsf{ker}(1 - U_t)$. Prove that the limit $\lim_{r\mapsto\infty} \frac{1}{r}\int_1^r U_t dt$ is equal to the orthogonal projection to H_0.

Exercise 2.2 (20 points). We say that an operator $A: H \longrightarrow H$ on a normed space is **Cesàro bounded** if there exists $C > 0$ such that $\|A_n\| < C$ for all n, where $A_n := \frac{1}{n}\sum_{i=0}^{n-1} A^i$. Prove that all unitary operators on a Hilbert space are Cesàro bounded, or find a counterexample.

Exercise 2.3 (10 points). Consider the shift operator operator $U(x_i) = x_{i+1}$ on a Hilbert space with orthonormal basis $x_1, x_2, \ldots.$. Prove that U is not a Koopman map of any dynamical system.

Exercise 2.4 (20 points). Consider the operator $U(x_i) = x_{i^3}$ on a Hilbert space with orthonormal basis $x_1, x_2, \ldots.$. Prove that U can be realized as a Koopman map of a dynamical system. Prove that this system is weakly mixing, or find a counterexample.

Exercise 2.5 (10 points). Let (M, μ, T) be an ergodic dynamical system, with $L^2(M)$ infinite-dimensional and $H \subset L^2(M)$ a finite-dimensional T-invariant subspace. Prove that none of the non-trivial eigenvalues of $T\big|_H$ are roots of unity or find a counterexample.

Exercise 2.6 (20 points). Let A be an orthogonal operator on a finite-dimensional space V. Prove that there exists a Hilbert space $H \supset V$, $H = L^2(M, \mu)$ and a Koopman operator $B : H \longrightarrow H$ of some dynamical system (M, μ, T), such that $B\big|_V = A$, or find a counterexample.

Exercise 2.7 (20 points). Let (M, μ, T) be a dynamical system, $\lambda \in \mathbb{C}$, and $H_\lambda \subset L^2(M, \mu)$ the eigenspace of $T : L^2(M, \mu) \longrightarrow L^2(M, \mu)$ associated with the eigenvalue λ. Prove that the set of bounded eigenfunctions is dense in H_λ.

Exercise 2.8 (20 points). Let A_1, A_2 be group automorphisms of a 2-dimensional torus $G = T^n$ defined by elements of $GL(n, \mathbb{Z})$. Prove that A_i preserve the usual measure on G. Assume that the action of A_1 and A_2 on G is ergodic. Prove that there is an isomorphism $\Phi : L^2(G) \longrightarrow L^2(G)$ mapping A_1 to A_2.

Exercise 2.9 (40 points). Let X, Y be metric spaces with a probability Borel measure, and $\phi : L^1(X) \longrightarrow L^1(Y)$ a linear operator. Suppose that $\int_Y \phi(f) = \int_X f$ for all $f \in L^1(X)$. Suppose, moreover, that for any family of functions $f_\alpha \in L^1(X)$ such that the supremum $\sup_\alpha(f_\alpha)$ is integrable, one has $\phi(\sup_\alpha(f_\alpha)) = \sup_\alpha \phi(f_\alpha)$. Prove that ϕ is induced by a measure-preserving map $\Phi : Y \longrightarrow X$.

Exercise 2.10 (40 points). Let X be a metric space with a probability Borel measure, and $\phi : L^2(X) \longrightarrow L^2(X)$ an isometric automorphism. Suppose that ϕ maps bounded L^2-functions to bounded ones, and for any two bounded measurable functions $f, g \in L^2(X)$, one has $\phi(fg) = \phi(f)\phi(g)$. Prove that ϕ is induced by a measure-preserving map $\Phi : X \longrightarrow X$.

3 Mixing, weak mixing, expanding, unique ergodicity

Exercise 3.1 (20 points). Let M be a closed Riemann surface of genus 2. Prove that there is no expansion maps from M to itself.

Exercise 3.2 (10 points). Let (M, μ, T) be a dynamical system. Prove that (M, μ, T^k) is weakly mixing if (M, μ, T) is weakly mixing. Prove that (M, μ, T) is weakly mixing if (M, μ, T^k) is weakly mixing.

Exercise 3.3 (10 points). Let (M, μ, T) be a dynamical system. Prove that (M, μ, T^k) is mixing if (M, μ, T) is mixing. Prove that (M, μ, T) is mixing if (M, μ, T^k) is mixing.

Exercise 3.4 (20 points). Let $A : T^n \longrightarrow T^n$ be given by a matrix $A \in GL(n, \mathbb{Z})$. Prove that A preserves the usual measure. Prove that A is mixing if and only if none of the eigenvalues of A are roots of unity.

Exercise 3.5 (20 points). Let $f : M \longrightarrow M$ be an isometry of a compact metric space. Assume that μ is a finite, uniquely ergodic measure on M. Prove that support of μ is M.

Exercise 3.6 (20 points). Let $f : M \longrightarrow M$ be an isometry of a non-compact metric space. Assume that μ is a finite, uniquely ergodic measure on M. Prove that support of μ is M, or find a counterexample

Exercise 3.7 (20 points). Let (M, μ, T) be a dynamical system. Assume that it is weak mixing. Prove that for all bounded measurable functions $\phi_1, ..., \phi_k$, one has

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} \int T^i(\phi_1) T^{2i}(\phi_2) ... T^{ki}(\phi_k) = \prod_{i=1}^{k} \int_M \phi_i.$$

Exercise 3.8 (20 points). Let M be a compact Riemannian manifold, and T an expanding map. Let $S = \{x_1, x_2, ...\}$ be the set of periodic points of T, and $p_1, p_2, ...$ their periods. Prove that the set of all x_i with $p_i < N$ is finite, for any given N. Prove that the measure $\sum_{i=1}^{\infty} \frac{1}{p_i} \delta_{x_i}$ is finite and T-invariant.

4 Hyperbolic geometry

Definition 4.1. Let $X \subset M$ be a subset of a metric space. Denote by $X(\varepsilon)$ the **ε-neighbourhood of** X, that is, the set of all points $y \in M$ satisfying $d(y, X) < \varepsilon$.

Definition 4.2. A subset Z of a complete Riemannian manifold is called **convex** if for any $x, y \in Z$, the geodesic interval connecting x to y lies in Z.

Exercise 4.1. a. (20 points) Let l be a geodesic in a hyperbolic plane $\mathbb{H}$. Prove that its ε-neighbourhood $l(\varepsilon)$ is convex for all $\varepsilon > 0$.

b. (10 points) Prove that an ε-neighbourhood of any convex subset of $\mathbb{H}$ is convex.

Definition 4.3. Reflection of a Poincaré plane $\mathbb{H}$ is a map $h \mapsto ghg^{-1}$ where $g : \mathbb{H} \longrightarrow \mathbb{H}$ is an isometry, and $h(x, y) = (-x, y)$ is the reflection of the upper half-plane with axis $x = 0$.

Exercise 4.2 (10 points). Let $f : \mathbb{H} \longrightarrow \mathbb{H}$ be a map which changes the orientation. Suppose that f has a fixed point on $\mathbb{H}$. Prove that it is a reflection.

Definition 4.4. Let Γ be a discrete group acting properly discontinously on a Riemannian manifold M. **Fundamental domain** of the Γ-action is a submanifold $D \subset M$ which intersects each Γ-orbit exactly once, outside of a measure 0 set.

Exercise 4.3. Let Γ be a group acting on a hyperbolic plane $\mathbb{H}$ properly discontinuously by isometries. Suppose that Γ is generated by reflections, and let S be the union of all geodesics $l \subset \mathbb{H}$ such that l is an axis of a reflection for some $\gamma \in \Gamma$.

a. (20 points) Prove that a connected component of $\mathbb{H} \backslash S$ is a fundamental domain of Γ.

b. (10 points) Prove that all these connected components are isometric.

Exercise 4.4. Let $D \subset \mathbb{H}$ be a triangle with vertices in absolute, and Γ the group generated by reflections across its edges.

a. (20 points) Prove that Γ acts on $\mathbb{H}$ properly discontinuously.

b. (10 points) Prove that the subgroup $\Gamma_0 \subset \Gamma$ of all isometries preserving the orientation acts on $\mathbb{H}$ freely.

Exercise 4.5 (20 points). Consider the natural action of $\Gamma = SL(2, \mathbb{Z})$ on $\mathbb{H}$. Prove that there exists a group $\tilde{\Gamma} \supset \Gamma$ generated by reflections such that $\Gamma \subset \tilde{\Gamma}$ is its index 2 normal subgroup.

Exercise 4.6 (20 points). Let $\mathbb{H}$ be a hyperbolic plane. Prove that there exists a constant C such that for any $x_1, ..., x_n \in \mathbb{H}$, the convex hull D of $x_1, ..., x_n$ lies in the C-neighbourhood of the union $\bigcup_i [y, x_i]$ of the geodesic intervals $[y, x_i]$, for any $y \in D$.

Exercise 4.7. Let $x_1, ..., x_n$ be points on a hyperbolic plane, D their convex hull, and ∂D its boundary.

a. (20 points) Prove that there exists a number $C_n > 0$ independent from the choice of the points $x_1, ..., x_n$, such that D lies in C_n-neighbourhood of ∂D.

b. (10 points) Prove that C_n can be chosen no greater than $A \log n$, where $A > 0$ is some constant independent on n.

Exercise 4.8 (20 points). Choose a left-invariant Riemannian metric on the group $G = SL(2, \mathbb{R})$, and let $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ act on G by $T(x) = AxA^{-1}$. Describe its stable and unstable foliations.

5 Lie groups, Hilbert spaces, measure theory

Exercise 5.1 (20 points). Consider a unitary action of the group S^1 on a complex Hilbert space H, and $H_1 \subset H$ a subspace generated by finite-dimensional subrepresentations. Prove that H is a closure of H_1.

Exercise 5.2 (20 points). Let $G = T^n$ be a torus, $A: \ G \longrightarrow G$ be given by a matrix $A \in GL(n, \mathbb{Z})$ and let $\hat{G}$ be the group of characters of G, that is, the group of homomorphisms $G \longrightarrow S^1$. Prove that A preserves the usual measure. Prove that A is ergodic if and only if the natural action of A on $\hat{G}$ has no periodic points.

Exercise 5.3 (40 points). Let G be $SO(p,q)$ with $p, q > 2$, and $\Gamma \subset G$ a lattice (that is, a subgroup such that G/Γ has finite measure). Prove that Γ is not commutative.

Exercise 5.4 (20 points). Let G be a Lie group. Recall that **Haar measure** on G is a left-invariant locally finite Borel measure. The group is called **unimodular** if any left-invariant measure is right-invariant. Prove that all nilpotent Lie groups are unimodular.

Exercise 5.5 (20 points). Let G be a compact Lie group, μ Haar measure, $g \in G$, and $L_g : \ G \longrightarrow G$, $L_g(x) := gx$ the left rotation map. Assume that (G, μ, L_g) is ergodic. Prove that G is commutative.

Exercise 5.6 (20 points). Let $G = SL(2, \mathbb{R})$ acts a Hilbert space H by isometries, giving a continuous map $G \longrightarrow O(H)$, where the group $O(H)$ is taken with the norm topology. Let $A_t := \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$, and suppose that $v \in H$ is A_t-invariant, for all $t \neq 0$. Prove that v is $G = SL(2, \mathbb{R})$-invariant.

Exercise 5.7 (20 points). Let Γ be a group acting on a hyperbolic space $\mathbb{H}^n$ properly and discontinuously, and D its fundamental domain. Assume that D is bounded. Prove that $\mathbb{H}^n/\Gamma$ is compact.

Definition 5.1. Let M be uncountable, second countable, complete metric space, and μ a Borel measure without points of positive measure. Then (M, μ) is called **Borel space**.

Exercise 5.8 (20 points). Prove that all probabilistic Borel spaces are isomorphic as spaces with measure.

Exercise 5.9 (20 points). Let $A \subset M$ be a subset of Borel space (M, μ) with positive measure. Prove that A is isomorphic (as a space with measure) to a Borel space.

Exercise 5.10 (10 points). Let (M, μ) be a metric space with Borel measure, $C > 0$, and $A \subset M$ a measurable subset such that for some $\varepsilon > 0$ and for all measurable $I \subset M$ of diameter $< \varepsilon$, one has $C\mu(A \cap I) \geqslant \mu(A)\mu(I)$. Prove that A or $M \backslash A$ is of measure 0.