Teoria Ergódica Diferenciável, assignment 7: Lie groups

Rules: This is a home assignment for October 18. Please bring me the solutions no later than October 25.

Exercise 7.1. Consider the group $O(1,1)$ of automorphisms of $\mathbb{R}^2$ preserving the scalar product of signature (1,1). Prove that $O(1,1)$ has at least 4 connected components.

Exercise 7.2. Let G be a topological group (a group which is a topological space with all group operations continuous). Prove that the connected component of G containing unit is a subgroup of G.

Exercise 7.3. Prove that the connected component of the unit of the group $O(1,1)$ is a group which is isomorphic to $\mathbb{R}$.

Exercise 7.4. Prove that the equation $x^2 - 2y^2 = 1$ has infinitely many integer solutions.

Hint. Use the previous exercise

Exercise 7.5. Prove that the group $SO(n)$ is connected.

Exercise 7.6. Let $\gamma \in O(p,q)$ be any automorphism of a vector space preserving a scalar product of signature (p,q). Prove that $|\det \gamma| = 1$.

Exercise 7.7. Construct an element $x \in SO(1,2)$ which is not diagonalizable over $\mathbb{C}$. Here $SO(1,2) \subset GL(3,\mathbb{R})$ is a group of matrices with $\det = 1$ preserving a scalar product of signature $(1,2)$.

Exercise 7.8. Let $\Psi : G \longrightarrow G_1$ be a homomorphism of connected Lie groups of the same dimension with $d\Psi$ surjective. Prove that Ψ is surjective, and its kernel is a discrete subgroup of G.