Teoria Ergódica Diferenciável, assignment 5: Krein-Milman theorem

Rules: This is a class assignment for September 20. Please try to write the solutions in class at September 20 and give them to your monitor Ermerson Rocha Araujo. No-one is penalized for failing to write the solutions, but being good at assignments would simplify getting good grades at your exams.

In this assignment, V is a Hausdorff topological vector space.

Definition 5.1. A **convex set** in a vector space V is a set S such that any two points $x, y \in S$ the set S contains an interval $I_{x,y} = \{tx + (1-t)y \mid 0 \leqslant t \leqslant 1\}$. **A convex cone** is a subset $S \subset V$ which is convex and preserved by homotheties $H_\lambda(x) = \lambda x$ for any $\lambda > 0$. **A convex hull** of a set $A \subset V$ is the smallest convex set containing A.

Exercise 5.1. Let $F \subset V = \mathbb{R}^n$ be a subset, and $\operatorname{Hull}(F)$ its convex hull. Prove that any point $x \in \operatorname{Hull}(F)$ can be represented as $x = \sum_{i=1}^{n+1} \lambda_i z_i$, where $0 \leqslant \lambda_i \leqslant 1$, $\sum_{i=1}^{n+1} \lambda_i = 1$, and $z_i \in F$.

Definition 5.2. Extreme point of a convex set K is a point $x \in K$ such that for any $a, b \in K$ and any $t \in [0,1]$, $ta + (1-t)b = x$ implies $a = b = x$.

Exercise 5.2. a. Let $A \subset \mathbb{R}^2$ be a compact convex subset. Prove that the set $E(A)$ of its extreme points is closed.

b. Find a compact convex subset $A \subset \mathbb{R}^3$ such that $E(A)$ is not closed.

Exercise 5.3. Let G be a group acting on a topological space, and $\mathcal{P}$ the space of probabilistic Borel measures. Using Radon-Nikodym theorem, prove that μ is an extremal point of $\mathcal{P}$ if and only if μ is ergodic.

Definition 5.3. Face of a convex set $A \subset V$ is a convex subset $F \subset A$ such that for all $x, y \in A$ whenever $\alpha x + (1-\alpha)y \in F$, $0 < \alpha < 1$, we have $x, y \in F$.

Exercise 5.4. Let $F_1 \subset F_2 \subset F_3$ be three convex sets, with F_1 being a face of F_2 and F_2 a face of F_3. Prove that F_1 is a face of F_3.

Exercise 5.5. Let $A \subset V$ be a convex set, and $\lambda : V \longrightarrow \mathbb{R}$ a linear map. Consider the set $F_\lambda := \{a \in A \mid \lambda(a) = \sup_{x \in A} \lambda(x)\}$. Prove that this set is a face of A.

Exercise 5.6. Let $x, y \in V$ be distinct points in a locally convex topological vector space. Prove that there exists a continuous linear functional $\lambda: V \longrightarrow \mathbb{R}$ such that $\lambda(x) \neq \lambda(y)$.

Hint. Use Hahn-Banach theorem.

Exercise 5.7. Let $W \subset V$ be a closed subspace in a locally convex topological vector space and $x \notin W$ a point in V. Prove that there exists a continuous functional $\lambda: V \longrightarrow \mathbb{R}$ such that $\lambda\Big|_W = 0$ and $\lambda(x) \neq 0$.

Hint. Apply Hahn-Banach theorem to extend an appropriate functional from the space generated by W and x to V.

Exercise 5.8. Prove that any compact convex subset $A \subset V$ which is not a point has a non-empty face $F \subsetneq A$.

Hint. Using Hahn-Banach theorem, find a continuous linear functional $\lambda: V \longrightarrow \mathbb{R}$, non-constant on A, and take $F_\lambda = \{a \in A \mid \lambda(a) = \sup_{x\in A} \lambda(x)\}$. Prove it is nonempty.

Exercise 5.9. Let $A \subset V$ be a compact convex subset. Prove that the set $E(A)$ of extreme points of A is non-empty.

Hint. Prove that a decreasing sequence of faces $F_1 \supsetneq F_2 \supsetneq F_3...$ satisfies $\bigcap_\alpha F_\alpha \neq 0$ and this intersection is also a face, and use the Zorn lemma and Exercise 5.8.

Exercise 5.10. (Krein-Milman theorem) Let $A \subset V$ be a compact convex subset a locally convex topological vector space. Let $A_1 \subset A$ be the closure of the convex hull of the set $E(A)$ of extreme points of A. Prove that $A = A_1$.

Hint. Suppose that $A \supsetneq A_1$. Using Hahn-Banach, find a functional λ which vanishes on A_1 and satisfies $\lambda(z) > 0$ for some $z \in A$. Take the face $F_\lambda = \{a \in A \mid \lambda(a) = \sup_{x\in A} \lambda(x)\}$ and find an extreme point in F_λ using the previous exercise.

Exercise 5.11. (Choquet theorem) Let $A \subset V$ be a compact convex set and $E := \overline{E(A)}$ the closure of set of its extreme points. Denote by $\mathcal{P}$ the set of all probabilistic Borel measures on A. Prove that for any $x \in A$ there exists a measure $d\mu$ on E such that $x = \int_{v\in E} v d\mu$.

Hint. Prove that the set of all x represented this way is compact, convex, and contains $E(A)$.

5.1 Supplement by Ermerson Rocha Araujo

Definition 5.4. A measure-preserving transformation f on a probability space $(X, \mathcal{X}, \mu)$ is called (strong) mixing if

$$\lim_n \mu(f^{-j}(A) \cap B) = \mu(A)\mu(B),$$

for any two measurable sets $A, B \in \mathcal{X}$.

Exercise 5.12. Show that an isometry of a compact metric space is **not** mixing for any invariant Borel measure whose support is not a single periodic orbit. In particular, circle rotations are not mixing.

Hint. Take three different points in the support of Borel measure.

Exercise 5.13. Show that a measure preserving system $(X, \mathcal{X}, \mu, f)$ is ergodic if and only if for all $A, B \in \mathcal{X}$ with $\mu(A), \mu(B) > 0$, there exists $n \in \mathbb{Z}$ for which $\mu(A \cap f^n(B)) > 0$.