Teoria Ergódica Diferenciável, assignment 4: Hahn-Banach theorem

Rules: This is a class assignment for September 13. Please try to write the solutions in class at September 13 and give them to your monitor Ermerson Rocha Araujo. No-one is penalized for failing to write the solutions, but being good at assignments would simplify getting good grades at your exams.

Definition 4.1. A **convex set** in a vector space V is a set S such that any two points $x, y \in S$ the set S contains an interval $I_{x,y} = \{tx + (1-t)y \ \mid \ 0 \leqslant t \leqslant 1\}$. **A convex cone** is a subset $S \subset V$ which is convex and preserved by homotheties $H_\lambda(x) = \lambda x$ for any $\lambda > 0$.

Remark 4.1. We shall need the following version of **Hahn-Banakh theorem.** Let V be a topological vector space over $\mathbb{R}$, $A \subset V$ - an open convex cone, $A \not\ni 0$, $W \subset V$ a closed subspace, and $\theta_W : W \longrightarrow \mathbb{R}$ a continuous linear functional which is positive on $W \cap A$. Then there exists a continuous linear functional $\theta : V \longrightarrow \mathbb{R}$ such that $\theta\big|_A > 0$, and $\theta\big|_W = \theta_W$.

Remark 4.2. Hahn-Banach theorem is essentially the only way to show that any topological vector space admits a non-zero continuous linear functional.

Exercise 4.1. Prove that Hahn-Banach theorem (Remark 4.1) is equivalent to the following statement. Let V be a topological vector space over $\mathbb{R}$, and $A \subset V$ an open convex cone, $A \not\ni 0$. Then there exists a hyperplane[1] $H \subset V$ not intersecting A ("absolute case of Hahn-Banach theorem"). Moreover, if a closed subspace $W \subset H$ contains a hyperplane H_W which does not intersect A, the hyperspace H can be chosen in such a way that $H \cap W = H_W$ ("relative case").

Exercise 4.2. Let $A \subset V$ be an open convex cone, $A \not\ni 0$, and $l \subset V$ be a line (a 1-dimensional vector subspace). Chose an orientation on l, $l = l^+ \cap 0 \cap l^-$ such that the negative part l^- does not intersect A.

a. Prove that the set $A + l^+$ of linear combinations $r + a$, $r \in l^+, a \in A$ is an open convex cone, $A + l^+ \not\ni 0$.

b. Suppose that the absolute case of the Hahn-Banach theorem is true. Prove that the hyperplane plane H can be chosen in such a way that $H \cap l = 0$.

Exercise 4.3. Let V be a topological vector space over $\mathbb{R}$, and $A \subset V$ an open convex cone, $A \not\ni 0$. Consider a closed subspace $W \subset H$, and let $H_W \subset W$ be a hyperplane which does not intersect A. Prove that the image of A in V/H_W is an open cone not intersecting W/H_W.

Exercise 4.4. Deduce the "relative" case of Hahn-Banach theorem from the "absolute" case.

Hint. Use the previous exercise and apply Exercise 4.2 to the image of A in V/H_W and $l = W/H_W \subset V/H_W$.

[1] **A hyperplane** is a closed codimension 1 subspace.

Exercise 4.5. Let $A \subset \mathbb{R}^2$ be an open, convex cone, not containing 0. Prove that $\mathbb{R}^2$ contains a hyperplane h not intersecting A.

Exercise 4.6. Let $V_0 \subset V$ be a hyperplane, and $A \subset V$ an open convex cone, $A \not\ni 0$. Suppose that in V_0 Hahn-Banach theorem is already proven, and V_0 contains a hyperplane H_0 not intersecting A.

a. Prove that the projection $\pi: V \longrightarrow V/H_0 = \mathbb{R}^2$ maps A to a convex, open cone $A' \subset \mathbb{R}^2$ such that $A' \not\ni 0$.

b. Prove that $V/H_0 = \mathbb{R}^2$ contains a hyperplane h such that $A' \cap h = 0$.

c. Show that V contains a hyperplane H such that $H \cap A = \emptyset$.

Hint. Use the previous exercise.

Exercise 4.7. a. Let $\{V_\alpha \subset V\}$ be a family of closed subspaces of a topological vector space V, linearly ordered by inclusion. Suppose that each V_α contains a hyperplane H_α, with $H_\alpha \cap V_{\alpha'}$ whenever $V_{\alpha'} \subset V_\alpha$. Prove that $\bigcup H_\alpha$ is a closed hyperplane in $\bigcup V_\alpha$

b. Prove Hahn-Banach theorem: for any a topological vector space V over $\mathbb{R}$, and an open convex cone $A \subset V$, $A \not\ni 0$, there exists a hyperplane $H \subset V$ not intersecting A

Hint. Use Zorn lemma to show that there exists a maximal subspace $V_1 \subset V$ such that the absolute case of Hahn-Banach theorem is true for the pair $V_1, A \cap V_1$. Prove that $V = V_1$ using Exercise 4.6.

Exercise 4.8 (*). Prove **Hahn-Banach separation theorem**. Let $A, A' \subset V$ be open, convex subsets, $A \cap A' = \emptyset$. Prove that there exists a continuous affine[2] map $f: V \longrightarrow \mathbb{R}$ such that $f\big|_A < 0$, $f\big|_{A'} > 0$.

Exercise 4.9. Let $L^\infty(\mathbb{Z})$ be the space of bounded functions on $\mathbb{Z}$, with topology given by the norm $|f| = \sup_{n\in\mathbb{Z}} f(n)$.

a. Fix $n \in \mathbb{Z}\backslash 0$, and let $R_n: L^\infty(\mathbb{Z}) \longrightarrow L^\infty(\mathbb{Z})$ be the **translation operator** mapping $f(x)$ to $g(x) = f(x-n)$. Denote by $A \subset L^\infty(\mathbb{Z})$ the cone of all functions which satisfy $f(x) > \varepsilon$ for some $\varepsilon > 0$. Prove that A is an open, convex cone, and for any $f \in A$, one has $f - R_n(f) \notin A$.

b. Prove that $L^\infty(\mathbb{Z})$ admits a continuous functional θ which is invariant under all translation operators and positive on A.

Hint. Use Hahn-Banach theorem applied to $V = L^\infty(\mathbb{Z})$, A as above, $\theta_W = 0$, and $W = \overline{\mathsf{im}(1 - R_1)}$.

Exercise 4.10 (*). A group G which admits a functional θ which is invariant under all translation operators and positive on a constant function $f = 1$ is called **amenable**. Prove that $\mathbb{Z}^n$ is amenable, for any $n = 2, 3, 4, ...$

Hint. Use the Hahn-Banach theorem in the same way.

[2] A map $f: V \longrightarrow \mathbb{R}$ is called **affine** if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for all $x, y \in V$.

4.1 Supplement by Ermerson Rocha Araujo

Definition 4.2. Let X a topological space and $f : X \to X$ a continuous map. We say that (f, X) is **topologically transitive** if for every pair U and V of non-empty open subsets of X there exists an integer $k \geqslant 0$ such that $f^k(U) \cap V \neq \emptyset$ (or equivalently $U \cap f^{-k}(V) \neq \emptyset$).

Exercise 4.11. Let X a metric space and $f : X \to X$ a homeomorphism. Suppose that f is topologically transitive and that X has an isolated point. Show that there is a **unique** f-invariant probability measure on (X, f).

Remark 4.3. Note that the transitivity hypothesis alone does not imply the existence of a unique invariant measure. For example, take the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

and consider the diffeomorphism of torus $f : \mathbb{T}^2 \to \mathbb{T}^2$ generated for A. We can prove that f is transitive, but we do not have unique f-invariant probability measure since $\overline{Per(f)} = \mathbb{T}^2$.

Exercise 4.12. Let $0 < \alpha < 1$ and consider the measure space $(\mathbb{R}/\mathbb{Z}; \mathcal{B}; \mu)$, where $\mathcal{B}$ is Borel σ-algebra and μ is the Lebesgue measure. Show that $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by $f(x) = -x + \alpha (\mathrm{mod}\ 1)$ is measure preserving but **not** ergodic.